Stochastic Positive Real Lemma and Synchronization over uncertain networks

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Abstract

In this paper, we prove the stochastic version of the Positive Real Lemma (PRL), to study the stability problem of nonlinear systems in Lur'e form with stochastic uncertainty. We study the mean square stability problem of systems in Lur'e form with stochastic parametric uncertainty affecting the linear part of the system dynamics. The stochastic PRL result is then used to study the problem of synchronization of coupled Lur'e systems, with stochastic interaction over the network, and provide a sufficiency condition for the synchronization problem. The sufficiency condition we provide for synchronization, is a function of nominal (mean) coupling Laplacian eigenvalues and the statistics of link uncertainty in the form of coefficient of dispersion (CoD). Under the assumption that the individual subsystems have identical dynamics, we show that the sufficiency condition is only a function of a single subsystem dynamics and mean network characteristics. This makes the sufficiency condition attractive from the point of view of computation for large size network systems. Interstingly, our results indicate that both the largest and the second smallest eigenvalue of the mean Laplacian play an important role in synchronization of complex dyanmics, characteristic to nonlinear systems. Simulation results for network of coupled oscillators with stochastic link uncertainty are presented to verify the developed theoretical framework.

I. Introduction

The study of network control systems is a topic that has received lots of attention among the research community lately. There is extensive literature on this topic involving both deterministic

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and stochastic network systems. Among various problems, the problem of characterizing the stability of estimator and controller design for linear time invariant (LTI) network systems in the presence of channel uncertainty is studied in [1], [2]. A similar problem involving nonlinear and linear time varying dynamics is studied in [3], [4], [5], [6]. The results in these papers discover fundamental limitations that arise in the design of stabilizing controller and estimator in the presence of channel uncertainty.

Another important problem in the study of network systems is that of synchronization of the individual systems interacting over a network which may have stochastic linkages. Applications in various fields motivate the presence of uncertainty in network systems. In electric power networks outage of transmission lines through mechanical failure or a malicious attack can be modeled as uncertainty. Synchronization with limited information or intermittent communication among individual agents, such as a network of neurons or a network of robots performing a synchronized activity, can also be modeled with a time varying uncertainty. Synchronization of systems over a network with stochastic links is an important area of research in power system dynamics [7]. Simplified power system models showing synchronization are being studied to gain insight into the effect of network topology on synchronization properties of dynamic power networks [8]. Effect of network topology and size on synchronization ability of complex networks is another important area of research [9]. The problem of synchronization in the presence of simple on-off or blinking interaction uncertainty has been an important area of research [10], [11], [12], [13]. Local synchronization for coupled maps is studied in [14], [15], which provides a measure for local synchronization. Synchronization over balanced neuron networks with random synaptic interconnections is studied [16]. Emergence of robust synchronized activity in networks with random interconnection weights has been studied [17]. Robustness of synchronization to small perturbations in system dynamics and noise has also been the focus of studies [18]. Hence, synchronization of nonlinear systems over a stochastic network is an important area of research with broad impact in various fields. But, synchronization of general nonlinear systems with uncertain interactions poses many chanllenges, and it may only be possible to arrive at local synchronizability [14], [15], or provide limitations on the synchronization gain [10] for the most general systems.

Passivity-based tools are used to study the stability problem for deterministic nonlinear network systems in [19], [20]. Synchronization of interconnected systems from input-output approach has been studied in [21] and shown to have applications in biological networks. These tools provide a systematic procedure for the analysis and synthesis of deterministic network systems. Synchronization of identical nonlinear systems over networks with stochastic link failures was previously studied by the authors in [10]. Without assumptions on nonlinearity, the authors were able to provide a necessary condition based on individual system characteristics like Lyapunov exponents and variance of link uncertainty. In this paper, under passivity assumptions on the system dynamics and nonlinearity, we aim to provide a sufficiency condition for synchronization of nonlinear systems over a network with stochastic links. Existing literature on the use of passivity based tools for analysis of stochastic systems assume additive uncertainty models [22], [23].

In this paper, we combine techniques from passivity theory and stochastic systems to provide a sufficient condition for the synchronization of uncertain network systems. This is achieved by proving a sufficient condition for stochastic stability of the error dynamics of these systems. To begin with, we prove a stochastic version of the Positive Real Lemma and provide an LMI-based verifiable sufficient condition for the mean square exponential stability of stochastic network. An important feature of the stochastic Positive Real Lemma is that the uncertainty enters multiplicatively in the system dynamics. This sufficient condition is then applied to study the problem of synchronization in network of Lur'e systems with uncertain linear interactions among the network subsystems. The sufficiency condition for mean square synchronization of the network is posed in terms of a sufficiency condition for mean square stability for a single subsystem of the network with parametric uncertainty and Laplacian eigenvalues of the mean network.

Usually in discrete-time consensus studies, where marginally stable systems are studied for convergence to average consensus, only the largest Laplacian eigenvalue plays a role in determining synchronizability. The second smallest eigenvalue is only utilized to study rate of convergence. On the other hand, in understanding synchronization of Lur'e systems over a stochastic network, both the largest and second smallest (Fiedler eigenvalue) eigenvalues play a crucial role. The Fiedler eigenvalue is well-known in graph theory literature as an indicator of algebraic connectivity of a graph. On the other hand, we believe that the largest Laplacian eigenvalue is an indicator of high degree of connectivity of some nodes termed as hub nodes. The sufficiency condition derived here can be solved using standard LMI techniques to study synchronization of the network, analyze effect of uncertainty in links or design network coupling. As the condition is posed in terms of a single subsystem it significantly reduces computational complexity associated with verifying the sufficiency condition. Thus, our proposed sufficiency condition is very attractive for the stability analysis of large scale uncertain network system.

Another interesting results proved in this paper is the dependence of the sufficiency condition on coefficient of dispersion of the network links. The coefficient of dispersion (CoD), defined as a ratio of variance to mean of a random variable indicates the amount of clustering behavior in the random variable. A CoD less than unity indicates patterns of occurence that are more regular. A CoD greater than unity indicates clusters of random occurences. Some real life networks display this behavior due to heavy tail distributions of uncertainties [24], [25]. The sufficiency condition derived shows that the synchronization of the network can be characterized by the mean square stability of a single subsystem with parametric uncertainty having CoD twice that of the maximum CoD for the uncertain links in the network.

The rest of the paper is structured as follows : In section II-A we formulate the general problem of stabilization of Lur'e systems with parametric uncertainty and prove the main results on the stochastic variant of Positive Real Lemma. The problem of synchronization is formulated and solved using the stochastic variant of PRL in section III-A. Simulation results are presented in section IV followed by conclusions in section V.

II. Stabilization of Uncertain Lur'e Systems

In this section, we first present the problem of stochastic stability of a Lur'e system with parametric uncertainty. The uncertainty is modeled as an independent identically distributed (i.i.d.) random processes. The main result of this section proves the stochastic version of the Positive Real Lemma.

A. Problem Formulation

We consider a Lur'e system, which has parametric uncertainty in the linear system dynamics. The uncertain system dynamics are described as follows:

$$x_{t+1} = A\left(\Xi(t)\right)x_t - B\phi(y_t,t) + v_t, \qquad y_t = Cx_t \tag{1}$$

where, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$, $\phi(y_t,t) \in \mathbb{R}^m$ is a nonlinear function, and, v_t is zeros mean additive noise vector with covariance R_v. $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ are the input and output matrices. The state matrix $A(\Xi(t)) \in \mathbb{R}^{n\times n}$ is uncertain, and the uncertainty is characterized by $\Xi(t) = [\xi_1(t), \ldots \xi_M(t)]^T$, where $\xi_i(t)$'s for $i \in \{1, \ldots, M\}$ are i.i.d. random processes with zero mean and variance σ_i^2, i.e., $E[\xi_i(t)] = 0$ and $E[\xi_i(t)^2] = \sigma_i^2$. The schematic of the system is depicted in Fig. 1. We make the following assumptions on the nonlinearity $\phi(y_t, t)$

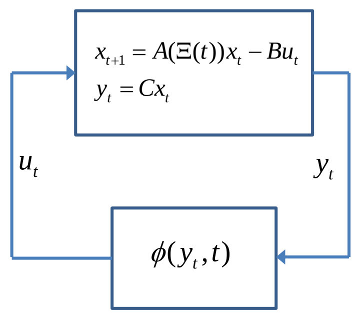


Fig. 1: Schematic of the system with parametric uncertainty.

Assumption 1: The nonlinearity $\phi(y_t, t)$ is a monotonic non-decreasing function of y_t such that, $\phi'(y_t, t)\,(y_t - D\phi(y_t, t)) > 0$.

The system, described by (1), encompasses a broad class of problems like stabilization under parametric uncertainty, control and observation of Lur'e system over uncertain channel [26], and network synchronization of Lur'e systems over uncertain links. Next, we state and prove a stochastic version of the Positive Real Lemma and successively use the result for network synchronization. The stochastic notion of stability that we use is the mean square stability [27] and is defined as follows:

Definition 2: The system in Eq. (1) is mean square exponentially stable if $\exists\, K > 0,$ and $0 < \beta < 1$, and $L > 0$, such that

$$E_\Xi \parallel x_t \parallel^2 \leq K\beta^t \parallel x_0 \parallel^2 + LR_v, \quad \forall\, x_0 \in \mathbb{R}^n. \tag{2}$$

where, x_t evolves according to (1).

Remark 3: The above definition of mean square stability holds for systems with additive noise. In case the additive noise is absent, the above definitions will reduce to the more familiar definition of mean square exponential stability [27], [26], [3], where $L = 0$.

B. Main Results

The following theorem is the stochastic version of the Positive Real Lemma providing sufficient condition for the mean square stability of the stochastic Lur'e system, described by (1).

Theorem 4: Let $\Sigma = D + D'$ and $A_T(\Xi(t)) = A(\Xi(t)) - B\Sigma^{-1}C$. Then the uncertain Lur'e system in (1) is mean square stable if -

1) there exist symmetric positive definite matrices P and R_P such that $\Sigma - B'PB > 0$ and,

$$P = E_{\Xi(t)}\left[A_T'(\Xi(t))PA_T(\Xi(t)) + A_T'(\Xi(t))PB(\Sigma - B'PB)^{-1}B'PA_T(\Xi(t))\right] + C'\Sigma^{-1}C + R_P \tag{3}$$

2) there exist symmetric positive definite matrices Q and R_Q such that $\Sigma - CQC' > 0$ and,

$$Q = E_{\Xi(t)}\left[A_T(\Xi(t))QA_T'(\Xi(t)) + A_T(\Xi(t))QC'(\Sigma - CQC')^{-1}CQA_T'(\Xi(t))\right] + R_Q + B'\Sigma^{-1}B \tag{4}$$

Proof: Please refer to the Appendix section for the proof. ■

The generalized version of stochastic Positive Real Lemma, as given by Theorem 4, is now specialized to the case of structured uncertainties. In particular, the structured uncertainties are assumed to be of the form $A(\Xi) = A + \sum_{i=1}^{M} \xi_i A_i$, where $\{\xi_i\}_{i=1}^{M}$ are zero mean i.i.d. random variables, the mean value having been incorporated in the deterministic part of the matrix given by A. The state and output equation for uncertain system becomes,

$$x_{t+1} = \left(A + \sum_{i=1}^{M} \xi_i A_i\right) x_t - B\phi(y_t, t) + v_t, \qquad y_t = Cx_t \tag{5}$$

The matrices A_i, adjoining to the uncertainties, could be pre-determined or could be designed depending on the problem. For instance, the results developed in [26] are for the scenario, where the matrix A_i is controller gain. The following Lemma simplifies the generalized stochastic PRL to study the mean square stability of system described by (5).

Lemma 5: The system, described in (5), would be mean square stable if there exists a symmetric matrix $P > 0$, such that $\Sigma - B'PB > 0$ and,

$$P = A_0'PA_0 + A_0'PB(\Sigma - B'PB)^{-1}B'PA_0 + C'\Sigma^{-1}C + R_P + \sum_{i=1}^{M} \sigma_i^2 \left(A_i'PA_i + A_i'PB(\Sigma - B'PB)^{-1}B'PA_i\right) \tag{6}$$

for some symmetric matrix $R_P > 0$ and $A_0 := A - B\Sigma^{-1}C$.

Proof: We substitute $\mathcal{A}(\Xi) = A + \sum_{i=1}^{M} \xi_i A_i$ in the (3) and utilize the fact ξ_i's are zero mean i.i.d. random variables with variance σ_i^2. We also $A_T(\Xi) = A + \sum_{i=1}^{M} \xi_i A_i - B\Sigma^{-1}C := A_0 + \sum_{i=1}^{M} \xi_i A_i$. Hence we get,

$$E_{\Xi(t)}\left[A_T'(\Xi(t))PA_T(\Xi(t))\right] = A_0'PA_0 + \sum_{i=1}^{M} \sigma_i^2 A_i'PA_i \tag{7}$$

Also we get,

$$\begin{aligned} E_{\Xi(t)}\left[A_T(\Xi(t))'PB(\Sigma - B'PB)^{-1}B'PA(\Xi(t))\right] &= A_0'PB(\Sigma - B'PB)^{-1}B'PA_0 \\ &\quad + \sum_{i=1}^{M} \sigma_i^2 A_i'PB(\Sigma - B'PB)^{-1}B'PA_i \end{aligned} \tag{8}$$

Combining equations (7) and (8) and substituting in (3) we get the desired result. ■

Corollary 6: The system, described in (5), would be mean square exponentially stable if there exists a symmetric matrix $Q > 0$, such that $\Sigma - CQC' > 0$ and,

$$\begin{aligned} Q &= A_0QA_0' + A_0QC'(\Sigma - CQC')^{-1}CQA_0' + B'\Sigma^{-1}B + R_Q \\ &\quad + \sum_{i=1}^{M} \sigma_i^2 \left(A_iQA_i' + A_iQC'(\Sigma - CQC')^{-1}CQA_i'\right) \end{aligned} \tag{9}$$

for some symmetric matrix $R > 0$ and $A_0 := A - B\Sigma^{-1}C$.

Proof: Corollary 6 follows from Theorem 4, Lemma 5 and duality. ■

III. Synchronization of Lur'e Systems with Uncertain Links

In this section, we apply the results developed in the previous section, in analyzing the problem of synchronization of Lur'e systems, coupled through uncertain links. We consider a set of linearly coupled systems in Lur'e form, where the interconnections between these systems, are uncertain in nature. In the subsequent section we derive a sufficiency condition for synchronization over a network, expressed in terms of uncertainty statistics and properties of the mean network, in particular the second smallest and largest eigenvalue of the interconnection Laplacian. The condition could be used to judge whether the coupled system with uncertainty could retain its synchronizability if the links binding the individual subsystems start to fail. Synchronization is achieved if the uncertainty variance satisfies prescribed bounds.

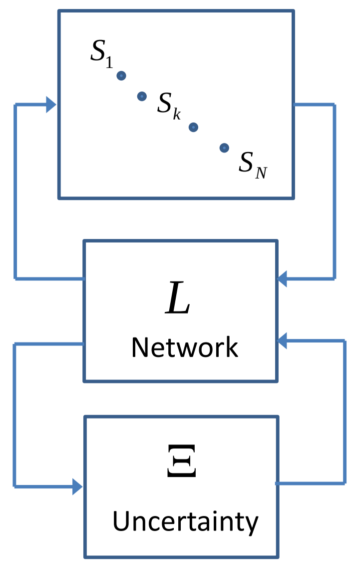


Fig. 2: Schematic of the interconnected system with uncertain links.

A. Formulation of Synchronization Problem

We consider a network of inter-connected systems in Lur'e form. The individual subsystems are described as follows:,

$$S_k := \begin{cases} x_{t+1}^k &= Ax_t^k - B\phi(y_t^k, t) \\ y_t^k &= Cx_t^k, \quad k = 1, \dots, N \end{cases} \tag{10}$$

where, $x^k \in \mathbb{R}^n$, and $y^k \in \mathbb{R}^m$ are the states and the output of k^{th} subsystem. The $\phi(y_n, n) \in \mathbb{R}^l$ is a nonlinear function. The state matrix $A \in \mathbb{R}^{n\times n}$ is the state matrix for k^{th} subsystem. $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ are the input and output matrices of the k^{th} subsystem. The inter-connected systems interacting with uncertainty through a network are depicted in Fig. 2. The non-linearity satisfies the following assumption,

Assumption 7: The nonlinearity $\phi_k(y_t^k, t) \in \mathbb{R}$ is globally Lipschitz monotonically nondecreasing function and C^1 function of $y_n^s \in \mathbb{R}$ that satisfies Assumption 1. Furthermore, it also satisfies the following condition,

$$\left(\phi\left(y_t^k\right) - \phi\left(y_t^j\right)\right)' \left(\left(y_t^k - y_t^j\right) - D_1\left(\phi(y_t^k) - \phi(y_t^j)\right)\right) > 0,$$

for any two systems S_k and S_j and some $\Sigma_1 = D_1 + D_1' > 0$.

The aforementioned assumption is essential for the synchronization of the network. Next, we consider coupled subsystems described by equation (10), that are linearly coupled, and analyze their synchronizability. The coupled system satisfies the following equation,

$$x_{t+1}^k = Ax_t^k - B\phi\left(y_t^k\right) + \sum_{j=1}^{N-1} \mu_{kj} G(y_t^j - y_t^k) + v_t, \qquad y_t^k = Cx_t^k, \quad k = 1, \dots, N \tag{11}$$

where, $\mu_{kj} \in \mathbb{R}$ represent the coupling link between subsystems S_k and S_j, $mu_{kk} = 0$ and $G \in \mathbb{R}^{n \times m}$.

Remark 8: The coupled system as described by (11) is the most general form of interaction possible between subsystems. The coupling between subsystems could be either in form of output feedback or state feedback. As the output and states of individual subsystems are related linearly so the form of coupling, as described by (11) includes both the output feedback and state feedback.

Next, we define the graph laplacian $L_g := [l_{ij}] \in \mathbb{R}^{N \times N}$ as following,

$$l_{ij} := \mu_{ij}, \ i \neq j, \quad l_{ii} := -\sum_{j, i \neq j} \mu_{ij}, \ i = 1, \dots N. \tag{12}$$

Next, all the states of the subsystems are combined to create the states of the coupled system. Finally the coupled system can be rewritten as,

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t - \tilde{B}\tilde{\phi}\left(\tilde{y}_t\right) - \left(L_g \otimes GC\right)\tilde{x}_t + v_t, \qquad \tilde{y}_t = \tilde{C}\tilde{x}_t, \tag{13}$$

where, $\otimes$ is the Kronecker product, I_n is an $n \times n$ Identity matrix and,

$$\tilde{A} := I_N \otimes A = \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{bmatrix}$$

We similarly define $\tilde{B} := I_N \otimes B$, $\tilde{C} := I_N \otimes C$, $\tilde{D}_1 := I_N \otimes D_1$ and $\tilde{\Sigma}_1 := \tilde{D}_1 + \tilde{D}_1' > 0$. We also define $\tilde{x}_t = [(x_t^1)' \dots (x_t^N)']'$, $\tilde{y}_t = [(y_t^1)' \dots (y_t^N)']'$, $\tilde{\phi}_t = [(\phi_t^1)' \dots (\phi_t^N)']'$.

B. Modeling Uncertain Links

We are now ready to study the problem of synchronization where the links of the graph are uncertain (i.e. entries of the Laplacian matrix are uncertain). Let

$$E_U = \{(i, j) | \text{the link } (i, j) \text{ is uncertain}, \ i > j\}$$

be the collection of uncertain links in the network. Hence, for links $(i,j) \in E_U$, we have $l_{ij} = \mu_{ij} + \xi_{ij},\ i \neq j$, where ξ_{ij} are zero mean i.i.d. random variables with variance σ_{ij}^2. If $(i,j) \notin E_U$ when we have $l_{ij} = \mu_{ij},\ i \neq j$ to be purely deterministic as in the previous subsection. This framework allows us to study synchronization for Lur'e type systems with a deterministic weighted Laplacian as a special case. Let $\Xi = \{\xi_{ij}\}_{(i,j)\in E_U}$. Then, the uncertain Laplacian $L_g(\Xi)$ will be given as,

$$L_g(\Xi) = L_m + \sum_{(i,j)\in E_U} \xi_{ij} L_{ij} \tag{14}$$

where L_m is the mean deterministic part of the Laplacian $L_g(\Xi)$, which may be written as $L_m = L_d + L_u$, where L_d is the part of the Laplacian constructed from μ_{ij} for purely deterministic edges $(i,j) \notin E_U$, while L_u is constructed from μ_{ij} for uncertain edges $(i,j) \in E_U$. We may also write $L_{ij} = \ell_{ij}\ell_{ij}'$ where $\ell_{ij} := [\ell_{ij}(1), \ldots, \ell_{ij}(N)]' \in \mathbb{R}^N$ is a column vector given by

$$\ell_{ij}(k) = \begin{cases} 0 & \text{if } k \neq i \neq j \\ 1 & \text{if } k = i \\ -1 & \text{if } k = j \end{cases}$$

We are interested in finding a sufficiency condition involving σ_{ij}^2 for $(i,j) \in E_U$, which would guarantee the mean square exponential synchronization. The coupled network of Lur'e system can be written as,

$$\tilde{x}_{t+1} = \Big(\tilde{A} - (L_g(\Xi) \otimes GC)\Big)\tilde{x}_n - \tilde{B}\tilde{\phi}(\tilde{y}_t) + v_t, \qquad \tilde{y}_t = \tilde{C}\tilde{x}_t \tag{15}$$

We would analyze the stochastic synchronization of system, described by (15). We start with following definition of mean square exponential synchronization.

Definition 9: The system, described by (15) is mean square exponentially synchronizing if there exists a $\beta < 1$, $K(\tilde{e}_0) > 0$, and, $L > 0$ such that,

$$E_\Xi \parallel x_t^k - x_t^j \parallel^2 \leq \bar{K}(\tilde{e}_0)\beta^t \parallel x_0^k - x_0^j \parallel^2 + LR_v, \quad \forall k,j \in [1,N] \tag{16}$$

where, $\tilde{e}_0$ is function of difference $\parallel x_t^i - x_t^\ell \parallel^2$ for $i,\ell \in [1,N]$ and $\bar{K}(0) = K$ for some constant K.

We now apply change of coordinates to decompose the system dynamics on and off the synchronization manifold. The synchronization manifold is given by $\mathbf{1} = [1,\ldots,1]'$. We show that the dynamics on the synchronization manifold is decoupled from the dynamics off the manifold and is essentially described by the dynamics of the individual system. The dynamics on the synchronization manifold itself could be stable, oscillatory, or complex. Let $L_m = V_m \Lambda_m V_m'$ where V_m is an orthonormal set of vectors given by $V_m = \left[\frac{\mathbf{1}}{\sqrt{N}}\; U_m\right]$, $\mathbf{1} = [1\ \cdots\ 1]'$ and U_m is orthonormal set of vectors also orthonormal to $\mathbf{1}$. Let $\tilde{z}_t = (V_m' \otimes I_n)\,\tilde{x}_t$. Multiplying (15) from the left by $V_m' \otimes I_n$ we get

$$\tilde{z}_{t+1} = \left(\tilde{A} - (V_m' L_g(\Xi) V_m \otimes GC)\right)\tilde{z}_t - \tilde{B}\tilde{\psi}\,(\tilde{w}_t) \tag{17}$$

where $\tilde{w}_t = \tilde{C}\tilde{z}_t$, and $\tilde{\psi}_t = (V_m' \otimes I_n)\,\tilde{\phi}\,(\tilde{y}_t)$. We can now write

$$\tilde{z}_t = \left[\begin{array}{cc} \bar{x}_t' & \hat{z}_t' \end{array}\right]', \quad \tilde{\psi}_t = \left[\begin{array}{cc} \bar{\phi}_t' & \hat{\psi}_t' \end{array}\right]' \tag{18}$$

where

$$\bar{x}_t := \frac{\mathbf{1}}{\sqrt{N}}\tilde{x}_t = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} x_t^k, \quad \hat{z}_t := (U_m' \otimes I_n)\,\tilde{x}_t \tag{19}$$

$$\bar{\phi}_t := \frac{\mathbf{1}}{\sqrt{N}}\tilde{\phi}\,(\tilde{y}_t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} \phi(y_t^k), \quad \hat{\psi}_t := (U_m' \otimes I_n)\,\tilde{\phi}\,(\tilde{y}_t) \tag{20}$$

Substituting (18) in (17) we get

$$\begin{aligned} \bar{x}_{t+1} &= A\bar{x}_t - B\bar{\phi}\,(\bar{y}_t) \\ \hat{z}_{t+1} &= \left(\hat{A} - (U_m' L_g(\Xi) U_m \otimes GC)\right)\hat{z}_t - \hat{B}\hat{\psi}\,(\hat{w}_t) \end{aligned} \tag{21}$$

where $\hat{w}_t = \hat{C}\hat{z}_t,\ \hat{A} := I_{N-1} \otimes A,\ \hat{B} := I_{N-1} \otimes B,\ \hat{C} := I_{N-1} \otimes C$, and $\hat{D}_1 := I_{N-1} \otimes D_1$. We now show that for the synchronization of system (15), we only need to stabilize $\hat{z}_t$ dynamics. The stability of the system with state $\hat{z}_t$, implies the synchronization of the actual coupled system. This feature is exploited to derive sufficiency condition for stochastic synchronization of the coupled system. In the following Lemma we show the connection between the stability of the described by (21) to the synchronization of the system described by (15).

Lemma 10: Mean square exponential stability of system described by (21) implies mean square exponential synchronization of the system (15) as given by Definition 9.

Please refer to the Appendix section of this paper for the proof. In the following subsection we will provide sufficiency conditions for the mean square exponential synchronization of (15) by proving sufficiency conditions for mean square exponential stability of (21). But first, we rewrite the equation (21) in a more suitable format. We note that $L_g(\Xi) = L_m + \sum_{E_U} \xi_{ij} L_{ij}$, and $L_m = V_m \Lambda_m V_m'$ where $V_m = \left[\frac{\mathbf{1}}{\sqrt{N}}\ U_m\right]$. Hence we have

$$U_m' L_g(\Xi) U_m = U_m' L_m U_m + \sum_{E_U} \xi_{ij} U_m' L_{ij} U_m := \hat{\Lambda}_m + \sum_{E_U} \xi_{ij} \hat{\ell}_{ij} \hat{\ell}_{ij}'$$

where $L_{ij} = \ell_{ij}\ell_{ij}'$, $\hat{\ell}_{ij} = U_m' \ell_{ij}$ and $\hat{\Lambda}_m := U_m' L_m U_m$ such that

$$\Lambda_m = V_m' L_m V_m = \begin{bmatrix} 0 & 0 \\ 0 & U_m' L_m U_m \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Lambda}_m \end{bmatrix}$$

Let $\mathcal{I} = \{\alpha_k\}_{k=1}^M$, $M = |E_U|$ be an indexing on uncertain edges in E_U. If index α_k corresponds to edge $(i,j) \in E_U$ then let $A_{\alpha_k} := U_m' L_{ij} U_m \otimes GC = \hat{\ell}_{ij}\hat{\ell}_{ij}' \otimes GC$. Thus we can write equation (21) as

$$\hat{z}_{t+1} = \left(\hat{A} - \hat{\Lambda}_m \otimes GC - \sum_{\alpha_k \in \mathcal{I}} \xi_{\alpha_k} A_{\alpha_k} \right) \hat{z}_t - \hat{B}\hat{\psi}_t(\hat{w}_t) \tag{22}$$

C. *Sufficiency Condition for Synchronization with Uncertain Links*

In previous subsection, we have shown that mean square exponential stability of (22) guarantees the mean square exponential synchronization of the coupled network of Lur'e system as given by (15). In the preceeding section, we have derived sufficiency condition for mean square stability of Lur'e system. In this subsection, we combine these two results to obtain sufficiency condition for mean square exponential synchronization of the network of Lur'e systems. The following Lemma provides the sufficiency condition for mean square synchronization.

Lemma 11: The system described by (15) is mean square exponential synchronizing if there exists a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{(N-1)n \times (N-1)n}$ such that,

$$\begin{aligned} \mathcal{P} &= (\hat{A}_0 - \Lambda_m \otimes GC)' \mathcal{P} (\hat{A} - \Lambda_m \otimes GC) + \sum_{\mathcal{I}} \sigma_{\alpha_k}^2 A_{\alpha_k}' \mathcal{P} A_{\alpha_k} \\ &+ (\hat{A}_0 - \Lambda_m \otimes GC)' \mathcal{P} \hat{B} \left(\hat{\Sigma}_1 - \hat{B}' \mathcal{P} \hat{B} \right)^{-1} \hat{B}' \mathcal{P} (\hat{A}_0 - \Lambda_m \otimes GC) \\ &\sum_{E_U} \sigma_{ij}^2 A_{\alpha_k}' \mathcal{P} \bar{B} \left(\hat{\Sigma}_1 - \hat{B}' \mathcal{P} \hat{B} \right)^{-1} \hat{B}' \mathcal{P} A_{\alpha_k} + \mathcal{R} \end{aligned} \tag{23}$$

and $\hat{\Sigma}_1 - \hat{B}'\mathcal{P}\hat{B} > 0$ for some symmetric matrix $\mathcal{R} > 0$ and $\hat{A}_0 := \hat{A} - \hat{B}\hat{\Sigma}_1^{-1}\hat{C} = I_{N-1} \otimes A_0$, $A_0 = A - B\Sigma_1^{-1}C$.

Proof: The proof follows from (15), (22), Lemma 10 and Theorem 4. ■

The above sufficiency condition is very difficult to verify for large size networks due to computational complexity associated with solving the Riccati equation. In particular the matrix $\mathcal{P}$ is of size $(N-1)*n \times (N-1)*n$ having $\frac{(N-1)^2n^2+(N-1)n}{2}$ variables to be determined. The number of variables increases quadratically with change in system dimension or size of network. In the following results, we exploit the identical nature of system dynamics to provide more conservative sufficient condition but with substantially reduced computational efforts. The sufficiency condition is based upon a single representative dynamical system modified using network characteristics, reducing number of variables to $\frac{n(n+1)}{2}$.

The new sufficient condition is also very insightful as it highlights the role played by the network property, in particular the second smallest and largest eigenvalues of the mean interconnection Laplacian, and the statistics of uncertainty in the sufficiency condition. The statistics of uncertainty is captured using the following definition of coefficient of dispersion.

Definition 12 (Coefficient of Dispersion): Let $\xi \in \mathbb{R}$ be a random variable with mean $\mu > 0$ and variance $\sigma^2 > 0$. Then, the coefficient of dispersion γ is defined as

$$\gamma := \frac{\sigma^2}{\mu}$$

To utilize the above definition in subsequent results we make an assumption on the system

Assumption 13: For all edges (i,j) in the network, the mean weights assigned are positive, i.e. $\mu_{ij} > 0$ for all (i,j). Furthermore, the coefficient of dispersion of each link is given by $\gamma_{ij} = \frac{\sigma_{ij}^2}{\mu_{ij}}$, and $\bar{\gamma} = \max_{\forall (i,j)}\{\gamma_{ij}\}$. This assumption simply states that the network connections are positively enforcing the coupling.

The following theorem provides a sufficiency condition for synchronization of the coupled systems based on the stability of a single modified system.

Theorem 14: The coupled system (15) is mean square exponentially synchronized if there exists a symmetric positive definite matrix $P > 0$ such that $\Sigma_1 - B'PB > 0$ and

$$P = (A_0 - \lambda_{sup}GC)'P(A_0 - \lambda_{sup}GC) + (A_0 - \lambda_{sup}GC)'PB(\Sigma_1 - B'PB)^{-1}B'P(A_0 - \lambda_{sup}GC)$$
$$2\bar{\gamma}\tau\lambda_{sup}\left(C'G'PGC + C'G'PB(\Sigma_1 - B'PB)^{-1}B'PGC\right) + C'\Sigma_1^{-1}C + R \tag{24}$$

for $R > 0$, $A_0 = A - B\Sigma_1^{-1}C$ and $\lambda_{sup} \in \{\lambda_2, \lambda_N\}$, where λ_N is the largest eigenvalue and λ_2 is the Fiedler eigenvalue, of the mean Laplacian. Furthermore, $\tau := \frac{\lambda_{N_u}}{\lambda_{N_u} + \lambda_{2_d}}$, where λ_{N_u} is the largest eigenvalue of the Laplacian for the uncertain graph L_u and λ_{2_d} is the second smallest eigenvalue of the purely deterministic Laplacian L_d.

Proof: Please refer to the Appendix section of this paper for the proof. ■

Remark 15: We have derived the sufficient condition for mean square exponential synchronization of coupled n-dimensional Lur'e systems by providing a sufficient condition for a single n-dimensional Lur'e system. This significantly reduces the computational load in determining the sufficient condition for synchronization of the coupled dynamics as the network size increases. It should be noted that the sufficient condition as provided in (24) can be written as a Riccati equation obtained for the stochastic Positive Real Lemma condition as derived in Theorem 4. Writing $\mu_c := \lambda_{sup}$ and $\sigma_c^2 := 2\bar{\gamma}\tau\lambda_{sup}$ we can write (24) as

$$P > (A_0 - \mu_c GC)' P (A_0 - \mu_c GC) + (A_0 - \mu_c GC) PB \left(\Sigma_1 - B'PB\right)^{-1} B'P(A_0 - \mu_c GC)$$
$$+ \sigma_c^2 \left(C'G'PGC + C'G'PB \left(\Sigma_1 - B'PB\right)^{-1} B'PGC\right) + C'\Sigma_1^{-1} C \tag{25}$$

Comparing with condition in Theorem 4, equation (25) is the sufficient condition for mean square stability of

$$x_{t+1} = \left(A - \xi GC\right) x_t - B\phi(y_t), \quad y_t = Cx_t \tag{26}$$

where ξ is an i.i.d. random variable with mean μ_c and variance σ_c^2. Thus the coefficient of dispersion of ξ is given by $\gamma_c = \frac{\sigma_c^2}{\mu_c} = 2\bar{\gamma}\tau$. Thus the synchronization of the coupled dynamics is guaranteed by the mean square exponential stabilization of an individual system, with parametric uncertainty in the state matrix multiplying the coupling matrix, having coefficient of dispersion twice that of the maximum coefficient of dispersion of the uncertain links of the network.

Remark 16 (Significance of τ): In Theorem 14, the factor $\tau := \frac{\lambda_{N_u}}{\lambda_{N_u} + \lambda_{2_d}}$ captures the effect of location and number of uncertain links, whereas $\bar{\gamma}$ captures the effect of intensity of the randomness in the links. It is clear that $0 < \tau \leq 1$. If the number of uncertain links ($|E_U|$) is sufficiently large, the graph formed by purely deterministic edge set may become disconnected. This will imply $\lambda_{2_d} = 0$, and, $\tau = 1$. Hence, for large number of uncertain links, λ_{N_u} is large while λ_{2_d} is small. In contrast, if a single link is uncertain, say $E_U = \{e_{kl}\}$, then $\tau = \frac{2\mu_{kl}}{2\mu_{kl} + \lambda_{2_d}}$. Hence, for a single uncertain link, the weight of the link has a degrading effect on the synchronization margin. The location of such an uncertain link will determine the value of $\lambda_{2_d} \leq \lambda_2$, thus degrading the synchronization margin. Based upon this observation, we can rank order individual links within a graph, with respect to their degradation of the synchronization margin, on the basis of location (λ_{2_d}), mean connectivity weight (μ), and the intensity of randomness given by CoD γ.

The condition for synchronization in Theorem 14 is provided in terms of both the second and the largest eigenvalues of the mean Laplacian. While the significance of the second smallest eigenvalue of the Laplacian in terms of graph connectivity is well-known in the literature, the significance of the largest eigenvalue of the Laplacian is not well documented. Next, we discuss the significance of the largest eigenvalue of Laplacian as it applies to our synchronization problem.

Remark 17 (Significance of Laplacian Eigenvalues): The second smallest eigenvalue, $\lambda_2 > 0$, of the graph Laplacian indicates algebraic connectivity of the graph. We observe from Theorem 14, as equation (25) is a quadratic in λ, there exist critical values of $\lambda_2(\lambda_N)$ for the given system parameters and CoD, below(above) which synchronization is not guaranteed, respectively. Hence, critical λ_2 indicates we require a minimum degree of connectivity within the network to accomplish synchronization. To understand the significance of λ_N, we look at the complement of the graph on the same set of nodes. We know from [28], sum of largest Laplacian eigenvalue of a graph and second smallest Laplacian eigenvalue of its complemet is constant. Thus, if λ_N is large the complementary graph has low algebraic connectivity. Thus, if we have hub nodes with high connectivity, then these nodes are sparsely connected in the complementary graph. Thus we interpret a high λ_N indicates a high presence of densely connected hub nodes. Therefore we conclude strong robustness property in synchronization is guaranteed for close to average connectivity of nodes as compared to isolated highly connected hub nodes.

IV. Simulation Results

We consider network of coupled oscillator system with linear coupling and stochastic uncertainty in their interactions. The dynamics of the individual oscillator is given by second order differential equation $\ddot{\theta}_k = \kappa \sin\theta_k$. We write the individual oscillator system in Lur'e form as follows

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{\kappa}{\pi} & 0 \end{pmatrix} x - \begin{pmatrix} 0 \\ -\kappa \end{pmatrix} \phi(y), \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

where we set $\kappa = 1$. The above system is then discretized using a zero order hold. We assume that the nonlinearity and the network interaction change only at discrete intervals and are constant during an interval. We choose the sampling time to be $T = 0.001$ seconds. The phase space dynamics of the discrete time uncoupled oscillator system in shown in Fig. 4. The phase space dynamics consists of two potential wells with periodic motion in each of the well. The oscillators are initialized so that two of the oscillators starts in one potential well and the other two in the second well. We study the synchronization of four coupled oscillators connected over the network as shown in Fig. 3 with output error coupling. We make the links connecting systems S_1 to S_3, S_2 to S_3 and S_2 to S_4 uncertain, shown as dashed red lines in Fig. 3

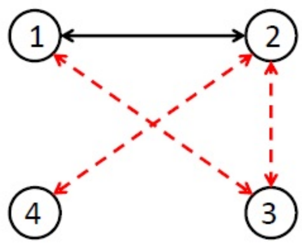


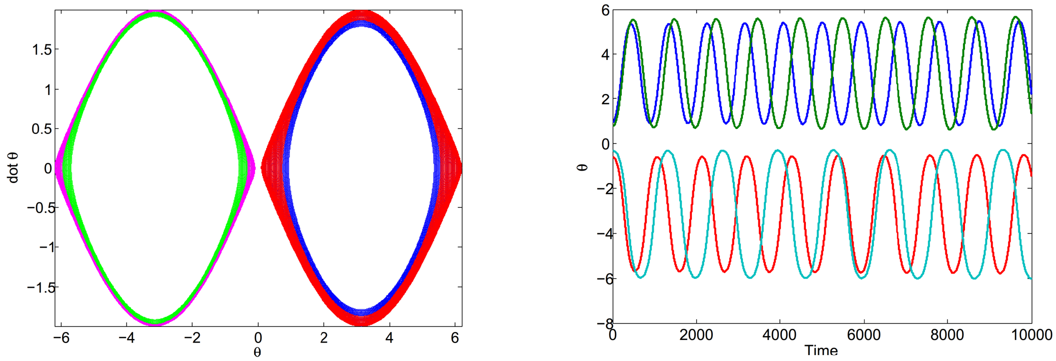
Fig. 3: Network connectivity of the systems with uncertain edges in red

The mean Laplacian for this network is given by

$$L_m = \begin{pmatrix} 1+\mu_{13} & -1 & -\mu_{13} & 0 \\ -1 & 1+\mu_{23}+\mu_{24} & -\mu_{23} & -\mu_{24} \\ -\mu_{13} & -\mu_{23} & \mu_{13}+\mu_{23} & 0 \\ 0 & -\mu_{24} & 0 & \mu_{24} \end{pmatrix}$$

. The uncertainties are modelled as i.i.d. Bernoulli uncertainties where the link connects with probability p and disconnects with probability $1-p$ for each uncertainty. The mean value for each connection is $\mu_{13} = \mu_{23} = \mu_{24} = p$ while the coefficient of variation $\bar{\gamma} = 1-p$. We choose the coupling matrix G as $G = g[1 \;\; 1]$ where we set $g = 0.002$.

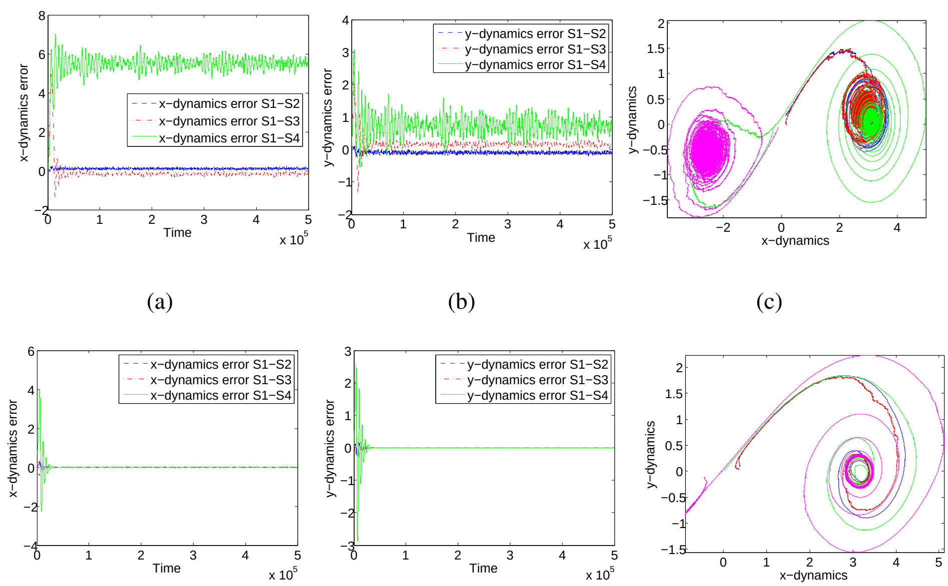
In Fig. 5, we show the simulation results for two different value of non-erasure probability p. We notice that for $p = 0.05$ the oscillators cannot synchronize and some are retained within their initial condition well. The minimum non-erasure probability required for synchronization, as predicted by solving the sufficiency condition is $p^* = 0.6$. This is obtained by solving the



(a) (b)

Fig. 4: a) State space dynamics of uncoupled oscillator; b) θ dynamics of four oscillators.

Riccati equation using LMI's and semi-definite programming (SDP) techniques. At $p = 0.6$ we see the systems synchronize and are able to pull the oscillators into a common well.



(a) (b) (c)

(d) (e) (f)

Fig. 5: (a) X-dynamics error for $p = 0.05$, (b) Y-dynamics error for $p = 0.05$, (c) Phase space plot for $p = 0.05$, (d) X-dynamics error for $p = 0.6$, (e) Y-dynamics error for $p = 0.6$, (f) Phase space plot for $p = 0.6$

V. CONCLUSIONS

In this paper we study the problem of synchronization of Lur'e systems over an uncertain network. This problem is presented as a special case of the problem of stabilization of Lur'e system with parametric uncertainty. Other special case of this problem include control of Lur'e system over an uncertain network which have been previously studied by the authors. These results are used to obtain some insightful results for the problem of synchronization over uncertain networks. We conclude that the sufficient condition for mean square exponential syncronization, of the coupled dynamics, is governed by mean square exponential stability of a representative system, with multiplicative parametric uncertainty in the state matrix. This uncertainty multiplies an output feedback based on the coupling matrix, that modifies the system dynamics. The uncertainty in the reprenstative system, has a CoD twice that of the maximum CoD in the network links and its mean is a function of the eigenvalue, of the mean network Laplacian.

As the sufficient condition is based on a single representative system, it is attractive from the point of view of computational complexity for large scale networks. This sufficient condition is solved as an LMI using Schur complement, similar to deterministic Positive Real Lemma. Furthermore, these results can be used to determine the maximum amount of dispersion tolerable within the network links. As expected we conclude that, if the randomness in the network links is highly clustered then it will be more difficult to synchronize the system. Another point of interest is that the synchronization of complex nonlinear systems, depends on the largest mean Laplacian eigenvalue along with the Fiedler eigenvalue, as opposed to just one for stable or marginally stable systems achieving consensus. This indicates that while, a certain minimum connectivity needs to be present to achieve synchronization, a high density of connections among nodes might be too much information for complex nonlinear system to synchronize under uncertainty.

VI. ACKNOWLEDGEMENTS

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APPENDIX

In the appendix we provide proofs for some of the important results we prove in the paper. We first provide the proof of Theorem 4.

Proof of Theorem 4: We show the conditions in Theorem 4 are indeed sufficient by constructing an appropriate Lyapunov function that guarantees mean square stability. We will prove the result in Theorem 4 for Case 1 and prove Case 2 as its dual. First, note that (3) holds if and only if

$$\begin{aligned} P = E_{\Xi(t)} &\left[(A'(\Xi(t))PB - C')(\Sigma - B'PB)^{-1}(C - B'PA(\Xi(t)))\right] \\ &+ E_{\Xi(t)}\left[A'(\Xi(t))PA(\Xi(t))\right] + R_P \end{aligned} \tag{27}$$

The equivalence of the two equations (3) and (27) is observed based on [29] (Proposition 12.1,1). Now consider the Lyapunov function $V(x_t) = x_t'Px_t$. Then, the condition for the system to be mean square stable is given by

$$\begin{aligned} E_{\Xi(t)}\left[V(x_{t+1}) - V(x_t)\right] =& x_t'\left(E_\Xi\left[A'(\Xi)PA(\Xi)\right] - P\right)x_t + 2x_t'E_\Xi[A'(\Xi)BP]\phi(y_t,t) \\ &+ \phi'(y_t,t)B'PB\phi(y_t,t) \end{aligned} \tag{28}$$

Substituting from (27) in (28) and applying algebraic manipulations as adopted in [30], we get

$$E_{\Xi(t)}\left[V(x_{t+1})\right] - V(x_t) = -\, x_t'R_Px_t - E_{\Xi(t)}\left[\zeta_t'\zeta_t\right] - 2\phi'(y_t,t)\left(y_t - D\phi(y_t,t)\right)$$

where $\zeta_t = \Sigma_P^{-\frac{1}{2}} \left(B'PA(\Xi(t)) - C\right) x_t - \Sigma_P^{\frac{1}{2}} \phi(y_t, t)$ and $\Sigma_P = \; (\Sigma - B'PB)$. From condition given in Assumption 1 we get $\phi'(y_t, t) \left(y_t - D\phi(y_t, t)\right) > 0$, which gives us,

$$E_\Xi \left[V(x_{t+1})) - V(x_t)\right] < -x_t' R x_t < 0$$

This implies mean square exponential stability of x_t and hence Case 1 is proved. Case 2 is now the dual to Case 1 by a simple argument as shown in [26]. ■

We now present proof of Lemma 10.

Proof of Lemma 10: From (19) we have

$$\| \hat{z}_t \|^2 = \tilde{x}_t' \left(U_m \otimes I_n\right) \left(U_m' \otimes I_n\right) \tilde{x}_t = \tilde{x}_t' \left(U_m U_m' \otimes I_n\right) \tilde{x}_t \tag{29}$$

Applying $U_m U_m' = V_m V_m' - \frac{\mathbf{1}}{\sqrt{N}} \frac{\mathbf{1}'}{\sqrt{N}} = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}'$ in (29) we get

$$\begin{aligned} \| \hat{z}_t \|^2 &= \tilde{x}_t' \left(\left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}' \right) \otimes I_n \right) \tilde{x}_t = \tilde{x}_t' \left(I_{Nn} - \left(\frac{1}{\sqrt{N}}\mathbf{1} \otimes \mathbf{I_n} \right) \left(\frac{1}{\sqrt{N}}\mathbf{1} \otimes \mathbf{I_n} \right)' \right) \tilde{x}_t \\ &= \tilde{x}_t' \tilde{x}_t - \bar{x}_t' \bar{x}_t = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j \neq i, j=1}^{N} \left(x_t^i - x_t^j\right)' \left(x_t^i - x_t^j\right) \end{aligned}$$

Now, mean square exponential stability of (21) implies there exists $K > 0$ and $0 < \beta < 1$ such that

$$\begin{aligned} E_\Xi \| \hat{z}_t \|^2 &\leq K\beta^t \| \hat{z}_0 \|^2, \\ E_\Xi \sum_{k=1}^{N} \sum_{j \neq k, j=1}^{N} \| x_t^k - x_t^j \|^2 &\leq K\beta^t \sum_{k=1}^{N} \sum_{j \neq k, j=1}^{N} \| x_0^k - x_0^j \|^2, \\ \Rightarrow \sum_{k=1}^{N} \sum_{j \neq k, j=1}^{N} E_\Xi \| x_t^k - x_t^j \|^2 &\leq K\beta^t \sum_{k=1}^{N} \sum_{j \neq k, j=1}^{N} \| x_0^k - x_0^j \|^2, \end{aligned}$$

This gives us the result,

$$E_\Xi \| x_t^k - x_t^l \|^2 \leq \bar{K}(\tilde{e}_0) \beta^t \| x_0^k - x_0^l \|^2 .$$

where $\bar{K}(\tilde{e}_0) := K \left(1 + \frac{\sum_{i=1, i\neq k}^{N} \sum_{j=1, j\neq i}^{N} \|x_0^i - x_0^j\|^2}{\|x_0^k - x_0^l\|^2} \right)$. ■

We now present proof of Theorem 14.

Proof of Theorem 14: We know mean square exponential synchronization is guaranteed by conditions in Lemma 11. Consider $\mathcal{P} = I_{N-1} \otimes P$ where $P > 0$ is a symmetric positive definite matrix that satisfies $\Sigma_1 - B'PB > 0$. This gives us $\hat{\Sigma_1} - \hat{B}'\mathcal{P}\hat{B} > 0$. Using this we write (23) as

$$I_{N-1} \otimes P > (\hat{A}_0 - \Lambda_m \otimes GC)'(I_{N-1} \otimes P)(\hat{A} - \Lambda_m \otimes GC) + \sum_{\mathcal{I}} \sigma_{\alpha_k}^2 A'_{\alpha_k}(I_{N-1} \otimes P)A_{\alpha_k}$$
$$+ (\hat{A}_0 - \Lambda_m \otimes GC)'(I_{N-1} \otimes P)\hat{B}\left(\hat{\Sigma}_1 - \hat{B}'(I_{N-1} \otimes P)\hat{B}\right)^{-1}\hat{B}'(I_{N-1} \otimes P)(\hat{A}_0 - \Lambda_m \otimes GC)$$
$$+ \sum_{\mathcal{I}} \sigma_{\alpha_k}^2 A'_{\alpha_k}(I_{N-1} \otimes P)\hat{B}\left(\hat{\Sigma}_1 - \hat{B}'(I_{N-1} \otimes P)\hat{B}\right)^{-1}\hat{B}'(I_{N-1} \otimes P)A_{\alpha_k}$$
$$+ I_{N-1} \otimes C'\Sigma_1^{-1}C \tag{30}$$

Since $A_{\alpha_k} = \hat{\ell}_{ij}\hat{\ell}'_{ij} \otimes GC$ we can write (30) as

$$I_{N-1} \otimes P > [A_0 - \lambda_j GC]'(I_{N-1} \otimes P)[A_0 - \lambda_j GC] + I_{N-1} \otimes C'\Sigma_1^{-1}C$$
$$+ [A_0 - \lambda_j GC]'\left(I_{N-1} \otimes (PB(\Sigma_1 - B'PB)^{-1}B'P)\right)[A_0 - \lambda_j GC]$$
$$+ \sum_{\mathcal{I}} \sigma_{\alpha_k}^2 (\hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \otimes GC)'\left(I_{N-1} \otimes (PB(\Sigma_1 - B'PB)^{-1}B'P)\right)(\hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \otimes GC)$$
$$+ \sum_{\mathcal{I}} \sigma_{\alpha_k}^2 (\hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \otimes GC)'(I_{N-1} \otimes P)(\hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \otimes GC) \tag{31}$$

where $[A_0 - \lambda_j GC] = (\hat{A}_0 - \Lambda_m \otimes GC)$. Inequality (31) can be further simplified as

$$I_{N-1} \otimes P > [A_0 - \lambda_j GC]'(I_{N-1} \otimes P)[A_0 - \lambda_j GC] + 2\sum_{\mathcal{I}} \sigma_{\alpha_k}^2 \hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \otimes C'G'PGC$$
$$+ [A_0 - \lambda_j GC]'\left(I_{N-1} \otimes (PB(\Sigma_1 - B'PB)^{-1}B'P)\right)[A_0 - \lambda_j GC]$$
$$+ 2\sum_{\mathcal{I}} \sigma_{\alpha_k}^2 \hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \otimes \left(C'G'PB(\Sigma_1 - B'PB)^{-1}B'PGC\right) + I_{N-1} \otimes C'\Sigma_1^{-1}C \tag{32}$$

We know that

$$\sum_{\mathcal{I}} \sigma_{\alpha_k}^2 \hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} = \sum_{\mathcal{I}} \gamma_{\alpha_k}\mu_{\alpha_k}\hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \leq \bar{\gamma}\sum_{\mathcal{I}} \mu_{\alpha_k}\hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} = \bar{\gamma}U'_m L_u U_m \tag{33}$$

We know that $L_m = L_u + L_d$. Thus if there exists $\tau \leq 1$ such that $L_u \leq \tau L_m$ then we must have $\frac{1-\tau}{\tau}L_u \leq L_d$. This is true if

$$\left(\frac{1-\tau}{\tau}\right)\lambda_{N_u} \leq \lambda_{2_d},$$

where λ_{N_u}, is the largest eigenvalue of the Laplacian L_u and λ_{2_d}, is the second smallest eigenvalue of the Laplacian L_d. We now choose $\tau = \frac{\lambda_{N_u}}{\lambda_{N_u}+\lambda_{2_d}}$ and applying $L_u \leq \tau L_m$ to (33) we obtain,

$$\sum_{\mathcal{I}} \sigma_{\alpha_k}^2 \hat{\ell}_{\alpha_k}\hat{\ell}'_{\alpha_k} \leq \bar{\gamma}U'_m(\tau L_m)U_m = \bar{\gamma}\tau\hat{\Lambda}_m \tag{34}$$

Now, substituting (34) in (32) a sufficient condition for inequality (32) to hold is given by

$$\begin{aligned}
I_{N-1} \otimes P > & \left[A_0 - \lambda_j GC\right]' \left(I_{N-1} \otimes P\right) \left[A_0 - \lambda_j GC\right] + I_{N-1} \otimes C' \Sigma_1^{-1} C \\
& + \left[A_0 - \lambda_j GC\right]' \left(I_{N-1} \otimes \left(PB \left(\Sigma_1 - B'PB\right)^{-1} B'P\right)\right) \left[A_0 - \lambda_j GC\right] \\
& + 2\bar{\gamma}\tau \hat{\Lambda}_m \otimes \left(C'G' \left(P + PB \left(\Sigma_1 - B'PB\right)^{-1} B'P\right) GC\right)
\end{aligned} \tag{35}$$

Equation (35) is essentially a block diagonal equation which gives the sufficient condition for mean square synchronization to be

$$\begin{aligned}
P > & (A_0 - \lambda_j GC)' P (A_0 - \lambda_j GC) + (A_0 - \lambda_j GC)' PB \left(\Sigma_1 - B'PB\right)^{-1} B'P(A_0 - \lambda_j GC) \\
& + 2\bar{\gamma}\tau\lambda_j C'G'PGC + 2\bar{\gamma}\tau\lambda_j C'G'PB \left(\Sigma_1 - B'PB\right)^{-1} B'PGC + C'\Sigma_1^{-1} C
\end{aligned} \tag{36}$$

for all non-zero eigenvalues λ_j of $\hat{\Lambda}_m$. Since (36) is a quadratic in the eigenavlues λ_j, it is sufficienct to study is the equations holds true for the extreme values of the set given by λ_2 and λ_N. This is easily seen by the following argument. Using Schur complement we can equivalently write (36) for a given λ_j and $C_1 = \sqrt{2\bar{\gamma}\tau} C$ as an LMI given by

$$M_1 + \lambda_j M_2 > 0 \tag{37}$$

where

$$M_1 = \begin{bmatrix} P - C'\Sigma_1^{-1} C & A_0'P & A_0'PB & 0 & 0 \\ PA_0 & P & 0 & 0 & 0 \\ B'PA_0 & 0 & \Sigma_1 - B'PB & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & -C'G'P & -C'G'PB & C_1'G'P & C_1'G'PB \\ -PGC & 0 & 0 & 0 & 0 \\ -B'PGC & 0 & 0 & 0 & 0 \\ PGC_1 & 0 & 0 & P & 0 \\ B'PGC_1 & 0 & 0 & 0 & \Sigma_1 - B'PB \end{bmatrix}.$$

Since this is a convex constraint in λ, if it is satisfied for any values of $\lambda_i, \lambda_j \in \{\lambda_2, \ldots, \lambda_N\}$, then (37) is true for any $\lambda = s\lambda_i + (1-s)\lambda_j$ for all $s \in [0, 1]$. Thus if we require (36) to hold for all eigenvalues of the mean Laplacian matrix, then it must hold for the extreme points of the set, i.e. $\lambda_{sup} \in \{\lambda_2, \lambda_N\}$. This proves the result. ■