Synchronization in Complex Network System With Uncertainty

Amit Diwadkar and Umesh Vaidya

Abstract— **In this paper, we study the problem of synchronization in a network of nonlinear systems with scalar dynamics. The nonlinear systems are connected over linear network with stochastic uncertainty in their interactions. We provide a sufficient condition for the synchronization of such network system expressed in terms of the parameters of the nonlinear scalar dynamics, the second and largest eigenvalues of the mean interconnection Laplacian, and the variance of the stochastic uncertainty. The provided sufficient condition is independent of network size thereby making it attractive for verification of synchronization in a large size network. The main contribution of this paper is to provide analytical characterization for the interplay of role played by the internal dynamics of the nonlinear systems, network topology, and uncertainty statistics for the network synchronization. We show that there exist important trade-offs between these various network parameters necessary to achieve synchronization. We provide simulation results in network system with internal dynamics modeling of agents moving in a double well potential function. The synchronization of network happens whereby the dynamics of the network system flip from one potential well to another at the backdrop of stochastic interaction uncertainty.**

I. Introduction

Synchronization in large scale network system is a fascinating problem that has attracted researcher attention from various disciplines of science and engineering. Synchronization is a ubiquitous phenomena in many engineering and naturally occurring systems. Examples include generators in electric power grid, communication network, sensor network, circadian clock, neural network in visual cortex in biological applications, and synchronization of fireflies [1]–[4]. In recent years, synchronization of systems over a network has gained significant importance in power system dynamics. Simplified power system models showing synchronization are being studied to gain insight into the effect of network topology on synchronization properties of dynamic power networks [5]. Effect of network topology and size on synchronization ability of complex networks is an important area of research [6]. Complex networks with certain desired properties like small average path between nodes, low clustering ability, existence of hub nodes among others, have been extensively studied over the past decade [7]–[12]. Understanding the effect of neighboring and long range communications on the ability and rate of synchronization, are important questions that will help understand molecular conformation [13]. Emergence of chimera states in synchronization is studied, where for mechanical systems coexistence of asynchronous states with synchronous states is demonstrated [14]. An aspect of network synchronization gaining attention is the effect of network topology and interconnection weights on robustness of synchronization properties [15].

Uncertainty plays an important role in many of these large scale network systems. Hence, the problem of robust synchronization in the presence of uncertainty is important for the design of robust network system. The presence of uncertainty in network systems can be motivated in various different ways. For example, in power network the uncertain parameters or outage of transmission lines could be source of uncertainty. Similarly, malicious attack on network links can be modeled as uncertainty. Synchronization with limited information or intermittent communication among individual agents can also be modeled as time varying uncertainty. In this paper, we address the problem of robust synchronization in large scale network systems. Existing literature on this problem has focused on the use of Lyapunov function-based techniques to provide condition for robust synchronization [16]. Similarly problem of synchronization in the presence of simple on-off or blinking interaction uncertainty is studied in [17]–[20]. Local synchronization for coupled maps is studied in [21], [22], which provides a measure for local synchronization. Our results differ from the existing work on this topic in a fundamental way as explained in the following discussion.

We consider a network of systems where the nodes in the network are dynamic agents with scalar nonlinear dynamics. These agents are assumed to interact linearly with other agents or nodes through the network Laplacian. The interaction among the network nodes is assumed to be stochastic. This research builds on our past work, where we have developed analytical framework for understanding fundamental limitations for stabilization and estimation of nonlinear systems over uncertain channels [23]–[27]. There are two main motivations of this research work which also form the main contributions of this paper. The first motivation was to provide scalable computational condition for the synchronization of large scale network systems. We exploit the identical nature of network agents dynamics to provide sufficient condition for synchronization which involves verifying a *scalar* inequality. This makes our condition independent of network size and hence attractive from the computational point of view for large scale network systems. The second motivation and contribution of this paper is to understand the *interplay* of internal agent dynamics, network

Financial support from National Science Foundation under award ECCS-1002053 and CNS-1329915 are gratefully acknowledged. A. Diwadkar and U. Vaidya are with the Department of Electrical & Computer Engineering, Iowa State University, Ames, IA 50011 diwadkar.amit@gmail.com, ugvaidya@iastate.edu

topology captured by graph Laplacian, and uncertainty statistics, and role played by each in network synchronization. We provide analytical relationship that help understand the *trade-off* between internal dynamics, network topology, and uncertainty statistics necessary for network synchronization. This analytical relationship will provide useful insight and comparison of robustness properties of complex network systems connected over varied network topologies such as scale-free, random, exponential, and small world.

The paper is organized as follows. The problem formulation and main results are presented in section II. The discussion on the interplay or role played by various network parameters is presented in section III. Simulation results are presented in section IV followed by conclusions in section V.

II. Synchronization in Dynamic Network with Uncertain Links

We consider a nonlinear network system where the first order dynamics of the individual network components is assumed to be of the form

$$x_{t+1}^k = ax_t^k - \phi(x_t^k) \quad k = 1, \ldots, N \tag{1}$$

where, $x^k \in \mathbb{R}$ are the states of k^{th} subsystem, and $a > 0$. The function $\phi : \mathbb{R} \to \mathbb{R}$ is a nonlinear function. The nonlinearity ϕ satisfies the following assumption,

Assumption 1: The nonlinearity $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous, globally Lipschitz function with global Lipschitz constant $\frac{2}{\delta}$.

The individual subsystem model is general enough to include system with steady state dynamics that could be stable, oscillatory, or chaotic in nature. The Jacobian of the nonlinear dynamics at the origin is given by $J = a - \frac{d\phi}{dx}(0) > a - \frac{2}{\delta}$. So for example, if $(a-1)\delta > 2$ we have $J > 1$ which would make the origin unstable. The negative feedback of the nonlinearity could then have a restoring effect and induce oscillatory or chaotic behavior.

We assume that the individual subsystems are linearly coupled over an undirected network given by a graph $G = (V, E)$, with node set V and edge set E with edge weights $\mu_{ij} \in \mathbb{R}^+$ for $i, j \in V$ and $e_{ij} \in E$. Let $E_U \subseteq E$ be a set of uncertain edges, and $E_D = E \backslash E_U$. The weights for the uncertain edges are $\mu_{ij} + \xi_{ij}$ for $e_{ij} \in E_U$, where μ_{ij} models the nominal edge weight and ξ_{ij} models the uncertainty. ξ_{ij} are random variables satisfying following statistics

$$E[\xi_{ij}] = 0, \qquad E[\xi_{ij}^2] = \sigma_{ij}^2.$$

Remark 1: The case corresponding to packet-drop or blinking [18] interaction uncertainty will be special case of the above defined more general random variable. For example, if $e_{ij} \in E_U$ is a blinking interconnection, it is modeled as a Bernoulli random variable with probability $p < 1$ to be on and $(1-p)$ to be off. Then μ_{ij} corresponding to e_{ij} is given by $\mu_{ij} = p$ and $\sigma_{ij}^2 = p(1-p)$.

If the coupling gain is $g > 0$ the individual agent dynamics of the coupled subsystem is given by,

$$\begin{aligned} x_{t+1}^k = & ax_t^k - \phi\left(x_t^k\right) + g \sum_{e_{kj} \in E_D} \mu_{kj}(x_t^j - x_t^k) \\ & + g \sum_{e_{ki} \in E_U} (\mu_{ki} + \xi_{ki})(x_t^i - x_t^k). \end{aligned}$$

We denote the purely deterministic graph Laplacian by $L_d := [\mu_{ij}], \forall e_{ij} \in E_D$ and mean Laplacian of the uncertain graph by $L_u := [\mu_{ij}], \forall e_{ij} \in E_U$. Furthermore, let $L_m = L_d + L_u$ be the mean Laplacian of the entire network, and $L_\xi = [\xi_{ij}], \forall e_{ij} \in E_U$. We combine the individual systems to create the network system $(\tilde{x}_t)$ written in compact form as,

$$\tilde{x}_{t+1} = \left(aI_N - g(L_m + L_\xi)\right)\tilde{x}_t - \tilde{\phi}\left(\tilde{x}_t\right), \tag{2}$$

where, I_N is the $N \times N$ identity matrix and $\tilde{x}_t = [(x_t^1)' \ldots (x_t^N)']'$, $\tilde{\phi}(\tilde{x}_t) = [(\phi^1)'(x_t^1) \ldots (\phi_t^N)'(x_t^N)]'$. There are four actors in Eq. (2) that we expect to play an important role in the synchronization. These are the internal dynamics of the network components captured by parameter a and nonlinearity ϕ, the deterministic graph Laplacian L_d, the uncertainty characteristics given by the variance σ_{ij} of the random variables, and the coupling gain g. Our objective is to understand the interplay of these actors to achieve synchronization in the network. Since the network is stochastic, we use following definition of mean square exponential (MSE) synchronization.

Definition 2 (MSE Synchronization): The system of equations described by (2) is mean square exponentially synchronizing, if there exists a $\beta < 1$ and $K(\tilde{e}_0) > 0$ such that,

$$E_\Xi \parallel x_t^k - x_t^j \parallel^2 \leq \bar{K}(\tilde{e}_0)\beta^t \parallel x_0^k - x_0^j \parallel^2, \quad \forall k, j \in [1, N] \tag{3}$$

where, $\Xi = \{\xi_{ij} \mid e_{ij} \in E_U\}$, $E_\Xi[\cdot]$ is expectation with respect to Ξ, $\tilde{e}_0$ is function of difference $\parallel x_0^i - x_0^\ell \parallel^2$ for $i, \ell \in \{1, \ldots, N\}$ and $\bar{K}(0) = K$ is a constant.

We introduce the following definition for the coefficient of dispersion to capture the statistics of uncertainty.

Definition 3 (Coefficient of Dispersion): Let $\xi \in \mathbb{R}$ be a random variable with mean $\mu > 0$ and variance $\sigma^2 > 0$. Then, the coefficient of dispersion γ is defined as

$$\gamma := \frac{\sigma^2}{\mu}$$

We make following assumption on the coupling constants and the coefficient of dispersion.

Assumption 4: For all edges (i, j) in the network, the mean weights assigned are positive, i.e. $\mu_{ij} > 0$ for all (i, j). Furthermore, the coefficient of dispersion of each link is given by $\gamma_{ij} = \frac{\sigma_{ij}^2}{\mu_{ij}}$, and $\bar{\gamma} = \max_{\xi_{ij}, e_{ij} \in E_U} \gamma_{ij}$. This assumption simply states that the network connections are positively enforcing the coupling.

The goal is to synchronize N first order systems over a network with mean graph Laplacian L_m having eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$, and maximum link uncertainty dispersion coefficient $\bar{\gamma}$. We will refer to the set $\{\lambda_2, \lambda_N\}$ as the boundary eigenvalues. Following theorem is the first main result of this paper.

Theorem 2: The network system (2) satisfying Assumptions 1 and 4 will achieve mean square synchronization (Definition 2) if there exists a positive scalar p such that

$$\delta > p > \alpha_0^2 p + \alpha_0^2 \frac{p^2}{\delta - p} + \frac{1}{\delta} \tag{4}$$

where, $\alpha_0^2 = (a_0 - \lambda_{sup} g)^2 + 2\lambda_{sup}\tau g^2$, $a_0 = a - \frac{1}{\delta}$,

$$\lambda_{sup} = \underset{\lambda \in \{\lambda_2, \lambda_N\}}{\operatorname{argmax}} \left| \lambda + \bar{\gamma} - \frac{a_0}{g} \right|,$$

and $\tau = \frac{\lambda_{N_u}}{\lambda_{N_u} + \lambda_{2_d}}$. λ_{N_u} is maximum eigenvalue of L_u, and λ_{2_d} is second smallest eigenvalue of L_d.

Proof: Please refer to the **Arxiv copy** of this paper [28] for the proof. ■

The condition in Theorem 2 requires one to find a positive scalar p which satisfies inequality (4). The following theorem provides condition based on system internal dynamics, network property, and uncertainty characteristics for network synchronization.

Theorem 3: The network system (2) satisfying Assumptions 1 and 4 will achieve mean square synchronization (Definition 2) if there exist $\delta > 1$, and,

$$\left(1 - \frac{1}{\delta}\right)^2 > \alpha_0^2 \tag{5}$$

where, $\alpha_0^2 = (a_0 - \lambda_{sup} g)^2 + 2\lambda_{sup}\tau g^2$, $a_0 = a - \frac{1}{\delta}$,

$$\lambda_{sup} = \underset{\lambda \in \{\lambda_2, \lambda_N\}}{\operatorname{argmax}} \left| \lambda + \bar{\gamma} - \frac{a_0}{g} \right|,$$

and $\tau = \frac{\lambda_{N_u}}{\lambda_{N_u} + \lambda_{2_d}}$. λ_{N_u} is maximum eigenvalue of L_u, and λ_{2_d} is second smallest eigenvalue of L_d.

Proof: From Theorem 2 and Eq. (4) we get the sufficient condition to be $\delta > p$, and,

$$\left(p - \frac{1}{\delta}\right)\left(\frac{1}{p} - \frac{1}{\delta}\right) > \alpha_0^2. \tag{6}$$

Since $\delta > p > \frac{1}{\delta}$ we must have $\delta > 1$. Furthermore, we have

$$\left(p - \frac{1}{\delta}\right)\left(\frac{1}{p} - \frac{1}{\delta}\right) \leq \left(1 - \frac{1}{\delta}\right)^2. \tag{7}$$

Using (7) we obtain that (5) is necessary for (4). Furthermore, if (5) is true then $p = 1$ satisfies the required condition (4) of Theorem 2. Hence (5) with $\delta > 1$ is equivalent to (4), proving Theorem 2 and Theorem 3 to be equivalent. ■

Remark 4: The sufficient condition for mean square exponential synchronization of N dimensional nonlinear network system (2) as derived in Theorem 2 is provided in terms of a scalar inequality instead of an N dimensional matrix inequality. This significantly reduces the computational load in determining the sufficient condition for synchronization of the coupled dynamics as the network size increases.

It should be noted that the sufficient condition as provided in Theorem 2, Eq. (4) is a Riccati equation in one dimension. Writing $\mu_c := \lambda_{sup}$ and $\sigma_c^2 := 2\bar{\gamma}\lambda_{sup}\tau$ we can write (4) as

$$p > \left((a_0 - \mu_c g)^2 + \sigma_c^2 g^2\right) p + \left((a_0 - \mu_c g)^2 + \sigma_c^2 g^2\right) \frac{p^2}{\delta - p} + \frac{1}{\delta}$$

For $q = \delta p$, this may be modified as

$$q > E_\xi \left[(a_0 - \xi g)^2 q + (a_0 - \xi g)^2 \frac{q^2}{\delta^2 - q} + 1 \right] \tag{8}$$

Using the condition of Bounded Real Lemma in [29] simplified for scalar systems, we notice that (8) is a sufficient condition for mean square stability of the 1D system given by

$$x_{t+1} = (a_0 - \xi g) x_t + \phi_0(x_t), \quad ||\phi_0(x)|| < \delta ||x|| \tag{9}$$

where ξ is an i.i.d. random variable with mean μ_c and variance σ_c^2. The coefficient of dispersion for ξ is given by $\gamma_c = \frac{\sigma_c^2}{\mu_c} = 2\bar{\gamma}\tau$. Thus the mean square exponential synchronization of coupled dynamics over a uncertain network is guaranteed if a one dimensional system with parametric uncertainty in the state matrix, having CoD twice that of the maximum CoD for uncertain links in the network, is robust to δ norm-bounded nonlinearity in the mean square sense. Thus, our sufficiency conditions in Theorems 2 and 3, indicate network robustness in a mean square sense towards coupling uncertainties.

We will now provide a condition relating the largest eigenvalue of a graph Laplacian to the eigenvalues of the Laplacian for the complementary graph which will be used for understanding the impact of the largest eigenvalue in synchronization. This condition is provided here for completion and the readers may refer [30] for the result and the proof.

Condition 5: Let $G \equiv (V, E)$ be a graph on $|V| = N$ nodes. Suppose that $\tilde{G} \equiv (V, \tilde{E})$ is the complement of G, such that $\tilde{G} = K_N \backslash G$, where K_N is the complete graph on N vertices. Let L_G and $L_{\tilde{G}}$ are the Laplacian matrices of G and $\tilde{G}$ with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ and $0 = \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_N$ respectively. Then we must have

$$\tilde{\lambda}_1 = \lambda_1 = 0, \quad \tilde{\lambda}_i = N - \lambda_{N-i+2}, \quad \forall\, i \in \{2, \ldots, N\} \tag{10}$$

III. Interplay of Internal Dynamics, Network Topology, and Uncertainty Characteristics

In this section, we discuss the role played by the internal dynamics, network topology, and uncertainty statistics for the synchronization of the network system. The internal dynamics is captured by parameters a and δ, the network topology is captured by the deterministic Laplacian, L_d, in particular the eigenvalues of the Laplacian, the uncertainty statistics is captured by CoD $\bar{\gamma}$ and τ captures the uncertainty location within the graph. We will make use of results from Theorems 2 and 3 to discover the interplay between the various parameters. We show that there are important trade-offs between the above mentioned parameters that impact synchronization over the network.

Remark 6 (Significance of τ): In Theorem 2, the factor $\tau := \frac{\lambda_{N_u}}{\lambda_{N_u} + \lambda_{2_d}}$ captures the effect of location and number of uncertain links, whereas $\bar{\gamma}$ captures the effect of intensity of the randomness in the links. It is clear that $0 < \tau \leq 1$. If the number of uncertain links ($|E_U|$) is sufficiently large, the graph formed by purely deterministic edge set may become disconnected. This will imply $\lambda_{2_d} = 0$, and, $\tau = 1$. Hence, for large number of uncertain links, λ_{N_u} is large while λ_{2_d} is small. In contrast, if a single link is uncertain, say $E_U = \{e_{kl}\}$, then $\tau = \frac{2\mu_{kl}}{2\mu_{kl}+\lambda_{2_d}}$. Hence, for a single uncertain link, the weight of the link has a degrading effect on the synchronization margin. The location of such an uncertain link will determine the value of $\lambda_{2_d} \leq \lambda_2$, thus degrading the synchronization margin. Based on this observation, we can rank order individual links within a graph, with respect to their degradation of synchronization, on the basis of location $(\lambda_{2_d}, \lambda_{N_u})$, mean connectivity weight (μ), and the intensity of randomness given by CoD γ.

Based on Theorem 3 we plot the figures in this section for a range of λ and $\bar{\gamma}$ values. For any given λ and $\bar{\gamma}$ value in the $\lambda - \bar{\gamma}$ space, using (5) we compute the quantity,

$$\chi(\lambda, \bar{\gamma}) = \max\{0, \left(1 - \frac{1}{\delta}\right)^2 - \alpha_0^2\}.$$

We plot $\chi(\lambda, \bar{\gamma})$ in the $\lambda - \bar{\gamma}$ space in this section to help us understand the interplay of various system and network parameters. We observe from Theorem 3 (condition (2) computed for no uncertainty, i.e. $\bar{\gamma} = 0$) that, there exists a critical value $\lambda_2^* = \frac{a_0}{g} - \frac{1}{g}\left(1 - \frac{1}{\delta}\right)$, below which synchronization is not guaranteed (colored white in Fig. 1, $\chi(\lambda, \bar{\gamma}) = 0$) for given system dynamics and coupling gain. The parameter values where synchronization is possible is marked in color, in Fig. 1. Recall that λ_2 is the measure of algebraic connectivity of mean network with larger value of λ_2 implies stronger algebraic connectivity of the network. Hence, we require a minimum degree of connectivity to accomplish synchronization. Similarly, it can be observed that, there is a critical value λ_N^* above which synchronization is not guaranteed. λ_N can be interpreted to quantify the connectivity of hub nodes. This is discussed in the following remark.

Remark 7 (Significance of λ_N): Let $G(V, E)$ be a graph on $|V| = N$ vertices with E as the edge set. Let $\tilde{G} = (V, \tilde{E})$ be the compliment of G. Then we know from Condition 5 equation (10), that $\lambda_N = N - \tilde{\lambda}_2$ where $\tilde{\lambda}_2$ is the second smallest eigenvalue of $\tilde{G}$. Thus if λ_N is large then $\tilde{\lambda}_2$ has to be small indicating lack of connectivity in the complementary graph. Thus if we have hub nodes with very high connectivity, then these nodes will be very sparsely connected in the complementary graph decreasing λ_2. If a hub node is connected to all the remaining nodes then $\lambda_N = N$ and $\tilde{\lambda}_2 = 0$ indicating that this node is disconnected in the complementary graph. Thus we interpret that a high λ_N indicates a high presence of densely connected hub nodes.

In Fig. 1, we plot two extreme cases for the system over a 1000 node network. We plot the results for a marginally stable system interconnected with a passive feedback nonlinearity. We observe that when the system is marginally stable only the largest Laplacian eigenvalue (λ_N) will determine synchronizability as long as $\lambda_2 > 0$ i.e. graph is connected. Thus for a given degree of nonlinearity (δ) as appropriate amount of coupling gain g will guarantee mean square synchronizability for any network, as long as the CoD of

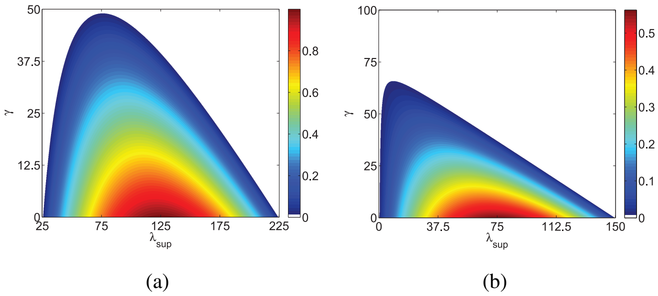


Fig. 1. $\chi(\lambda, \bar{\gamma})$ in $\lambda - \bar{\gamma}$ parameter space indicating region of synchronization(colored)/desynchronization(white) a) $a = 1.25, g = 0.01$, and $\delta = \infty$; b) $a = 1, g = 0.01$, and $\delta = 4$

the link uncertainty lies below a threshold $\bar{\gamma}_{max} = \frac{1}{g}\left(1 - \frac{1}{\delta}\right)$, which is obtained from Theorem 3. Fig. 1(b) on the other hand, studies the extreme case of linear systems obtained by setting $\delta = \infty$. For this case, we observe that for a given value of $\bar{\gamma}$, we require a certain minimum algebraic connectivity (λ_2) to ensure mean square synchronization. Furthermore, λ_N cannot exceed a bound indicating limitations on high connectivity of the network. These bounds on λ_2 and λ_N are worst case bounds for the given system dynamics as characterized by a and coupling gain g.

We now choose a base set of values $a = 1.25$, $g = 0.01$ and $\delta = 4$ and plot the results in $\lambda - \bar{\gamma}$ space as shown in Fig. 2. We vary the parameters for the system dynamics, nonlinearity bounds and the coupling gain to observe their interaction with the critical Laplacian eigenvalues λ_2^* and λ_N^* and the maximum allowable dispersion in uncertainty values $\bar{\gamma}$. Following conclusions are drawn from these parameter variations.

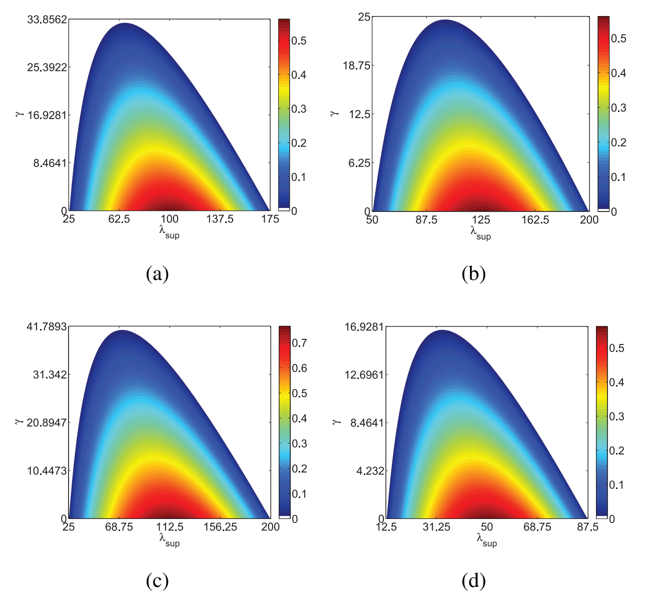


Fig. 2. $\chi(\lambda, \bar{\gamma})$ in $\lambda - \bar{\gamma}$ parameter space indicating region of synchronization(colored)/desynchronization(white) a) $a = 1.25, g = 0.01$, and $\delta = 4$; b) $a = 1.5, g = 0.01$, and $\delta = 4$; c) $a = 1.25, g = 0.01$, and $\delta = 8$; d) $a = 1.25, g = 0.02$, and $\delta = 4$

A. Effect of Internal Stability (a)

The effect of internal instability (a) as observed from Fig 1 and Fig. 2, is to demand minimum level of connectivity (λ_2) within a network, to guarantee synchronization. The increase in internal instability for higher value of a will require improved network connectivity for synchronization and hence increase in critical value λ_2^*. The maximum eigenvalue λ_N^* is also proportional to a indicating high instability in systems aids synchronization if there is high level of communication. On the other hand as is expected, increase in instability, guarantees synchronization only under lower uncertainty (smaller $\bar{\gamma}$). In Fig. 2(b), we show the region of synchronization/desynchronization in $\lambda - \bar{\gamma}$ space for parameter values of $a = 1.5, \delta = 4$, and $g = 0.01$. The effect of instability on maximum allowable uncertainty $\bar{\gamma}$ as we decrease the amount of nonlinearity within a system.

B. Effect of Nonlinearity Bound (δ)

The effect of nonlinearity bound δ is to demand smaller range of nodal interconnection density (small λ_N) while maintaining certain average number of interconnections (minimum required λ_2). The parameter δ is inversely proportional to sector of nonlinearity i.e., increase in δ leads to smaller sector of nonlinearity. In Fig. 2(c), we show the region of synchronization/desynchronization in $\lambda_2 - \bar{\gamma}$ space for parameter values of $a = 1.25, \delta = 8$, and $g = 0.01$. λ_2^* is independent of δ while λ_N^* is directly proportional to δ. Thus we conclude that very high level of communication through few hub nodes is harmful for synchronization of highly nonlinear systems with large sectors. As uncertainty increases the level of communication has to drop in order for the network to synchronize.

C. Effect of Coupling Gain (g)

The effect of coupling gain g is to allow synchronization under sparse network connectivity (small λ_2) with low levels of network uncertainty ($\bar{\gamma}$). Increase in gain g leads to decrease in region where synchronization occurs in $\lambda - \bar{\gamma}$ parameter space. λ_2^*, λ_N^* and $\bar{\gamma}_{max}$ are inversely proportional to the value of g indicating that high gain is detrimental to synchronization of networks with significant number of densely connected nodes (large λ) as opposed to majority of nodes with small average connectivity (small λ_N). Furthermore, a large gain might aid synchronization of sparsely connected networks (small λ_2). In Fig. 2(d), we show the region of synchronization/desynchronization in $\lambda_2 - \bar{\gamma}$ space for parameter values of $a = 1.25, \delta = 4$, and $g = 0.02$.

Remark 8: The analytical characterization for synchronization condition in Theorems 2 and 3 makes it possible to provide precise trade-off between the various network parameters. We refer the readers to [28] for more details discussions on these trade-offs and its applications to understand robustness properties of synchronization using various complex network topologies.

IV. Simulation results

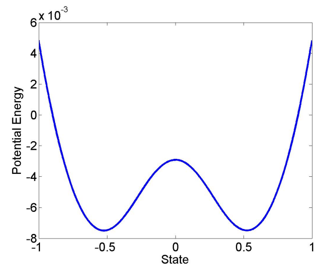
We consider the following 1D system

$$x_{t+1} = ax_t - \phi(x_t) \tag{11}$$

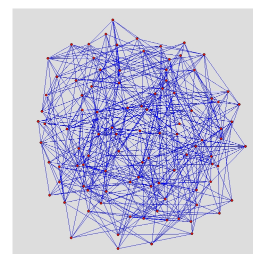
where $a = 1.125$ and $\delta = 4$. Here $\phi(x)$ is given by

$$\begin{aligned}\phi(x_t) = \tfrac{\mathrm{sgn}(x_t)}{4}\big(\tfrac{1+m_2}{2}(|x_t| - \varepsilon) \\ + \tfrac{1}{2}\left((1-m_2)^2(|x_t|-\varepsilon)^2 + 4m_2\varepsilon^2\right)\big)\end{aligned}$$

where $m_2 = \frac{1}{1+10\varepsilon^{0.1}}$ and $\varepsilon = 0.3$. The internal dynamics of the system as described by Eq. (11) consists of double well potential Fig. 3(a) with an unstable equilibrium point at the origin and two stable equilibrium points at $x^* = \pm\varepsilon\left(\frac{a-1}{a-2} + \frac{m_2(a-1)}{m_2(a-1)-1}\right) = \pm 0.5237$. So with no network coupling i.e., $g = 0$, the internal dynamics of the agents will converge to the the positive equilibrium point $x^* > 0$ for positive initial conditions. Similarly if the initial condition is negative, the systems converge to the negative equilibrium point $x^* < 0$.



(a)

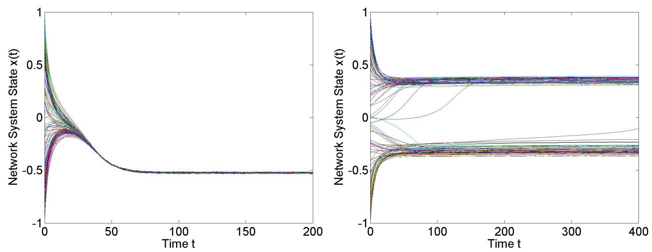


(b)

Fig. 3. a) Double-well potential function for system dynamics, b) Small World Network graph

We couple this system over a network of 100 nodes, generated as a random network with the Small World property (using the Pajek [31] network visualization software), Fig. 3(b). The coupling gain for this system is $g = 0.005$. The mean Laplacian of the network is considered to be a standard Laplacian with unit weight. Thus for all links e_{ij} connecting nodes i and j, $\mu_{ij} = 1$. This network has $\lambda_N = 52.55$ and $\lambda_2 = 26.23$. The uncertainty ξ in the network link weights, is chosen as a uniform variable with variance $\sigma^2 = \frac{1}{48}$, such that both these eigenvalues satisfy the required condition from Theorem 3. The CoD of the link uncertainty $\bar{\gamma} = \frac{\sigma^2}{\mu} = \frac{1}{48}$. We plot the result in Fig. 4(a), which shows that the systems synchronize to the equilibrium point.

For systems over the network with identical parameters to those in the previous case and identical link noise variance, if the coupling gain is decreased to $g = 0.001$ which does not satisfy the requirement of Theorem 3, we observe that the system is not able to synchronize (Fig. 4(b)), and the points with positive initial conditions converge to the positive equilibrium and visa-versa for the points with negative initial conditions. We do see some movement to the opposing equilibrium point for initial conditions very close to the origin. This may be possible due to the connectivity of



(a) (b)

Fig. 4. a) Time evolution of systems over a 100 node Small World network, $\bar{\gamma} = \frac{1}{48}$, $g = 0.005$, b) Time evolution of systems over a 100 node Small World network, $\bar{\gamma} = \frac{1}{48}$, $g = 0.001$

such nodes to nodes in the other well and the fact that they are extremely close to the origin, which creates a dividing barrier between the two potential wells, such that any small computational inaccuracy or stochasticity allows it to overcome the potential barrier.

V. Conclusions

We studied the problem of synchronization in complex network systems in the presence of stochastic interaction uncertainty among network nodes. We exploited identical nature of internal node dynamics to provide sufficient condition for the network synchronization. The unique feature of the sufficient condition is that it is independent of network size. This makes the sufficient condition attractive from the computational point of view for large scale network system. Furthermore, the sufficient condition provides useful insight into the interplay between the internal dynamics of the network nodes, network interconnection topology, and uncertainty statistics and their roles in network synchronization. Our results will help understand and compare various complex network topologies for a given internal nodal dynamics.

References

[1] S. H. Strogatz and I. Stewart, "Coupled oscillators and biological synchronization," *Scientific American*, vol. 269, no. 6, pp. 102–109, 1993.

[2] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, "The kuramoto model: A simple paradigm for synchronization phenomena," *Rev. Mod. Phys.*, vol. 77, pp. 137–185, Apr 2005.

[3] T. Danino, O. Mondragon-Palomino, L. Tsimring, and J. Hasty, "A synchronized quorum of genetic clocks," *Nature*, vol. 463, pp. 326–330, 2010.

[4] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proceedings of the National Academy of Sciences*, 2013.

[5] M. Rohden, A. Sorge, D. Witthaut, and M. Timme, "Impact of network topology on synchrony of oscillatory power grids," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 24, no. 1, 2014.

[6] J. Stout, M. Whiteway, E. Ott, M. Girvan, and T. M. Antonsen, "Local synchronization in complex networks of coupled oscillators," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 21, no. 2, pp. –, 2011.

[7] P. Erdös and A. Rényi, "On random graphs, I," *Publicationes Mathematicae (Debrecen)*, vol. 6, pp. 290–297, 1959.

[8] D. J. Watts and S. H. Strogatz, "Collective dynamics of'small-world'networks.," *Nature*, vol. 393, no. 6684, pp. 409–410, 1998.

[9] L. A. N. Amaral, A. Scala, M. Barthlmy, and H. E. Stanley, "Classes of small-world networks," *Proceedings of the National Academy of Sciences*, vol. 97, no. 21, pp. 11149–11152, 2000.

[10] A.-L. Barabsi and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[11] L. M. Pecora, F. Sorrentino, A. M. Hagerstrom, T. E. Murphy, and R. Roy, "Cluster synchronization and isolated desynchronization in complex networks with symmetries," *Nature Communnications*, vol. 5, 2014.

[12] L. Becks and H. Arndt *Nature Communications*, vol. 4, pp. 1359–1367, 2013.

[13] I. Mezic, "On the dynamics of molecular conformation," *Proceedings of the National Academy of Sciences*, vol. 103, no. 20, pp. 7542–7547, 2006.

[14] E. A. Martens, S. Thutupalli, A. Fourrire, and O. Hallatschek, "Chimera states in mechanical oscillator networks," *Proceedings of the National Academy of Sciences*, vol. 110, no. 26, pp. 10563–10567, 2013.

[15] T. Nishikawa and A. E. Motter, "Network synchronization landscape reveals compensatory structures, quantization, and the positive effect of negative interactions," *Proceedings of the National Academy of Sciences*, vol. 107, no. 23, pp. 10342–10347, 2010.

[16] X. Wang and G.Chen, "Synchronization in scale-free dynamical networks: Robustness and fragility," *IEEE Transactions on Circuit and Systems-I: Fundamental theory and applications*, vol. 49, no. 1, pp. 54–62, 2002.

[17] A. Diwadkar and U. Vaidya, "Robust synchronization in network systems with link failure uncertainty," in *Proceedings of IEEE Control and Decision Conference*, (Orlando, FL), pp. 6235–6330, 2010.

[18] I. Belykh, V. Belykh, and M. Hasler, "Blinking models and synchronization in small-world networks with a time-varying coupling," *Physica D*, vol. 195, pp. 188–206, 2004.

[19] M. Hasler, V. Belykh, and I. Belykh, "Dynamics of stochastically blinking systems. part i: Finite time properties," *SIAM Journal on Applied Dynamical Systems*, vol. 12, no. 2, pp. 1007–1030, 2013.

[20] M. Hasler, V. Belykh, and I. Belykh, "Dynamics of stochastically blinking systems. part ii: Asymptotic properties," *SIAM Journal on Applied Dynamical Systems*, vol. 12, no. 2, pp. 1031–1084, 2013.

[21] L. Wenlian, F. M. Atay, and J. J, "Synchronization of discrete-time dynamical networks with time-varying coupling," *SIAM Journal of Mathematical Analysis*, vol. 39, no. 4, pp. 1231–1259, 2007.

[22] L. Wenlian, F. M. Atay, and J. J, "Chaos synchronization in networks of coupled maps with time-varying topologies," *European Physics Journal B*, vol. 63, pp. 399–406, 2008.

[23] A. Diwadkar and U. Vaidya, "Limitation on nonlinear observation over erasure channel," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 454–459, 2013.

[24] A. Diwadkar and U. Vaidya, "Stabilization of linear time varying systems over uncertain channel," *International Journal of Robust and Nonlinear Control*, pp. 1205–1220, 2014.

[25] A. Diwadkar, S. Dasgupta, and U. Vaidya, "Control of systems in Lure form over erasure channels," *International Journal of Robust and Nonlinear Control*, 2014, http://dx.doi.org/10.1002/rnc.3231.

[26] U. Vaidya and N. Elia, "Limitation on nonlinear stabilization over packet-drop channels: Scalar case," *Systems and Control Letters*, pp. 959–966, 2012.

[27] U. Vaidya and N. Elia, "Limitation on nonlinear stabilization over erasure channel," in *Proceedings of IEEE Control and Decision Conference*, (Atlanta, GA), pp. 7551–7556, 2010.

[28] A. Diwadkar and U. Vaidya, "Limitations and tradeoff in synchronization of large-scale stochastic networks," 2014, http://arxiv.org/abs/1409.3249. Pre-print.

[29] W. Haddad and D. Bernstein, "Explicit construction of quadratic Lyapunov functions for the small gain theorem, positivity, circle and Popov theorems and their application to robust stability. Part II: Discrete-time theory," *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 249–265, 1994.

[30] R. Merris, "Laplacian matrices of graphs: a survey," *Linear Algebra and its Applications*, vol. 197-198, no. 0, pp. 143 – 176, 1994.

[31] V. Batagelj and A. Mrvar, "Pajek - Program for Large Network Analysis." http://vlado.fmf.uni-lj.si/pub/networks/pajek/.