Optimal stabilization using Lyapunov measure

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Abstract—The focus of the paper is on the computation of optimal feedback stabilizing control for discrete time control system. We use Lyapunov measure, dual to the Lyapunov function, for the design of optimal stabilizing feedback controller. The linear Perron-Frobenius operator is used to pose the optimal stabilization problem as a infinite dimensional linear program. Finite dimensional approximation of the linear program is obtained using set oriented numerical methods. Simulation results for the optimal stabilization of periodic orbit in one dimensional logistic map are presented.

I. INTRODUCTION

Stability analysis and stabilization of nonlinear systems are two of the most important and extensively studied problems in control theory. Lyapunov function and Lyapunov function based methods have played an important role in providing solutions to these problems. In particular, the Lyapunov function is used for stability analysis and the control Lyapunov function (CLF) is used for the design of stabilizing feedback controllers. Another problem which is extensively studied in controls literature is the optimal control problem (OCP). Optimal control for the OCP can be obtained from the solution of the Hamilton Jacobi Bellman (HJB) equation. Under the additional assumption of detectability and stabilizability of nonlinear system, the optimal cost function if positive can also be used as control Lyapunov function. This establishes the connection between stability (Lyapunov function) and optimality (HJB equation). The HJB equation is a nonlinear partial difference equation and hence, difficult to solve analytically and one has to resort to approximate numerical schemes for its solution. We review some of the literature particularly relevant to this paper on the approximation of HJB equation and OCP.

In [1], an adaptive space discretization scheme is used to obtain the solution of deterministic and stochastic discrete time HJB (dynamic programming) equation. Optimal cost function is obtained as a fixed point solution of a linear dynamic programming operator. In [2], [3], cell mapping approach is used to construct approximate numerical solutions for deterministic and stochastic optimal control problems. In [4], [5], set oriented numerical methods are used to under-estimate the optimal one-step cost for transition between different state-space discretizations in the context of optimal control and optimal stabilization. This allows to represent the minimal cost control problem as one of finding the minimum cost path to reach the invariant set on a graph with edge costs derived from the under-estimation procedure. Dijkstra’s algorithm is used to construct an approximate solution to the HJB equation. In [6], [7], [8] solutions to deterministic optimal control problems are proposed by casting them as infinite dimensional linear programs. Approximate solution to the infinite dimensional linear program is then obtained using finite dimensional approximation of the linear programming problems or using sequence of LMI relaxation.

In this paper we propose the use of Lyapunov measure for the optimal stabilization of nonlinear systems. Lyapunov measure is introduced in [9], to study weaker set wise notion of almost everywhere stability and is shown to be dual to the Lyapunov function. Existence of Lyapunov measure guarantees stability from almost every with respect to Lebesgue measure initial conditions in the phase space. Control Lyapunov measure is introduced in [10] to provide Lyapunov measure based framework for the stabilization of nonlinear systems. In this paper we extend this framework for the optimal stabilization of nonlinear systems. One of the main highlights and contributions of this paper is that finite dimensional deterministic optimal stabilizing control is obtained as the solution of finite linear program.

This paper is organized as follows. In section II, we provide a brief overview of some of the key results from [9],[11],[10] for stability analysis and stabilization of nonlinear systems using Lyapunov measure. The framework for optimal stabilization using Lyapunov measure and transfer operators is posed as an infinite dimensional linear program in section III. A computational approach based on set oriented numerical methods in proposed for the finite dimensional approximation of the linear program in section IV. Simulation results for optimal stabilization of periodic orbit in one dimensional logistic map are presented in section V, followed by conclusion and discussion in section VI.

II. LYAPUNOV MEASURE, STABILITY AND STABILIZATION

In [9], [11], [10], Lyapunov measure is introduced for stability verification and for stabilizing controller design of an invariant set in nonlinear dynamical systems. Stability and stabilization problems of an attractor set $A$ for a nonlinear system $T : X \rightarrow X$, where $X \subset \mathbb{R}^n$ is compact, were studied using a weaker notion of almost everywhere stability.

Definition 1 (Attractor set): A closed set $A$ is said to be $T$ invariant if $T(A) = A$. A closed $T$ invariant set $A$ is called an attractor set if there exists a local neighborhood $V$ of $A$ such that $T(V) \subset V$.

We use following definition of almost everywhere (a.e.) stability with geometric decay of the attractor set in this paper.
**Definition 2:** An attractor set $A$ is said to be a.e. stable with geometric decay with respect to measure $m$ if given $\delta > 0$, there exists $K(\delta) < \infty$ and $\beta < 1$ such that

$$m\{x \in A^c : T^n(x) \in B\} < K(\delta)\beta^n$$

for all set $B \subset X \setminus U_\delta$, where $U_\delta$ is the $\delta$ neighborhood of an attractor set $A$ and $\delta$ is the complement of the invariant set.

**Remark 3:** In the subsequent section we use the notation $m$ for the Lebesgue measure, $m_\delta$ for the Lebesgue measure supported on set $S$ and $U_\delta$ for the $\delta$ neighborhood of the attractor set $A$ for a given $\delta > 0$.

This weaker notion of a.e. stability is studied using a Linear transfer operator called as Perron-Frobenius (P-F) operator. P-F operator is used to study the propagation of sets or the measure supported on the sets. For any given continuous mapping $T : X \to X$, linear Perron-Frobenius (P-F) operator, denoted by $\mathbb{P}_T : \mathcal{M}(X) \to \mathcal{M}(X)$ is given by

$$\mathbb{P}_T[\mu](B) = \int_X \chi_B(T(x))d\mu(x)$$

where $\mathcal{M}(X)$ is the vector space of all measures supported on $X$, $\chi_B(x)$ is the indicator function supported on the set $B \subset \mathcal{B}(X)$, the Borel sigma-algebra of $X$. For more details on the P-F operator refer to [12]. Since the stability property of an attractor set in definition (2) is stated in terms of the transient behavior of the system on the complement of the attractor set $A^c$, we define sub-stochastic Markov operator as a restriction of the P-F operator on $A^c$ as follows:

$$\mathbb{P}_T^c[\mu](B) := \int_X \chi_B(T(x))d\mu(x)$$

for any set $B \in \mathcal{B}(A^c)$ and $\mu \in \mathcal{M}(A^c)$. Necessary and sufficient condition for almost everywhere uniform stability of an invariant set $A$ with respect to any finite measure $m$ were obtained in the form of existence of the positive solution, Lyapunov measure, to the following **Lyapunov measure equation**:

$$\gamma \mathbb{P}_T^c \mu(B) - \mu(B) = -m(B)$$

The precise theorem for stability as proved in [11] is:

**Theorem 4:** The attractor set $A$ for the dynamical system $T : X \to X$ is a.e. stable with geometric decay with respect to measure $m$ if and only if there exists a non-negative measure $\mu$ which is finite on $\mathcal{B}(X \setminus U_\delta)$ and satisfies

$$\gamma \mathbb{P}_T^c \mu(B) - \mu(B) = -m(B)$$

for every set $B \subset X \setminus U_\delta$ and for some $\gamma > 1$. Measure $m$ is absolutely continuous with respect to Lyapunov measure $\mu$.

Stability of the attractor set with respect to Lebesgue almost every initial condition starting from a given set $S$ can be studied by taking $m = m_S$ in the Lyapunov measure equation. In [13], set oriented numerical approach is used for the finite dimensional approximation of the Lyapunov measure $\mu$. This finite dimensional approximation leads to further weaker notion of stability, which is referred to as **coarse stability**. Unlike almost everywhere stability, coarse stability of an invariant set allows for the existence of stable dynamics in the complement of an invariant set however the domain of attraction of the stable dynamics is strictly smaller than the size of the partition used in the finite dimensional approximation.

In [10], Lyapunov measure is used for the design of stabilizing feedback controller. For the stabilization problem we consider the control dynamical system of the form

$$x_{n+1} = T(x_n, u_n)$$

where $x_n \in X$ and $u_n \in U$ is the state space and control space respectively. The objective is to design feedback controller $u_n = K(x_n)$ to stabilize the invariant set $A$, which is assumed to be locally stable and hence forms an attractor set. The stabilization problem is solved using Lyapunov measure by extending the P-F operator formalism to the control dynamical system as follows. We define the feedback control mapping $C : X \to Y := X \times U$ as $C(x) = (x, K(x))$. Using the definition of feedback mapping $C$, we write the feedback control system as

$$x_{n+1} = T(x_n, K(x_n)) = T \circ C(x_n)$$

With the system mapping $T : Y \to X$ and the control mapping $C : X \to Y$, we can associate Perron-Frobenius operators $\mathbb{P}_T : \mathcal{M}(Y) \to \mathcal{M}(X)$ and $\mathbb{P}_C : \mathcal{M}(X) \to \mathcal{M}(Y)$ respectively, and are defined as follows

$$\mathbb{P}_T[\theta](B) = \int_Y \chi_B(T(y))d\theta(y)$$

$$\mathbb{P}_C[\mu](D) = \int_D f(a|x)dm(a)d\mu(x)$$

where $\theta \in \mathcal{M}(Y), \mu \in \mathcal{M}(X)$ and $B \subset X, D \subset Y$. $f(a|x)$ is the conditional probability density function and is introduced to incorporate the particular form of feedback controller mapping $C(x) = (x, K(x))$. The advantage of writing the feedback control dynamical system as the composition of two maps $T : Y \to X$ and $C : X \to Y$ is that the P-F operator for the composition $T \circ C : X \to X$ can be written as a product of $\mathbb{P}_T$ and $\mathbb{P}_C$ as follows (refer to [10]):

$$\mathbb{P}_{T \circ C} = \mathbb{P}_T \cdot \mathbb{P}_C : \mathcal{M}(X) \to \mathcal{M}(X).$$

In [10], control Lyapunov measure is introduced for the stabilization of nonlinear systems. Control Lyapunov measure is defined as any non-negative measure $\bar{\mu} \in \mathcal{M}(A^c)$, which is finite on $\mathcal{B}(X \setminus U_\delta)$ and satisfies

$$\mathbb{P}_T^c \bar{\mu}(B) < \beta \bar{\mu}(B)$$

for every set $B \subset X \setminus U_\delta$ and $\beta < 1$. Operators $\mathbb{P}_T^c$ and $\mathbb{P}_C$ are the restrictions of the P-F operator $\mathbb{P}_T$ and $\mathbb{P}_C$ to the complement of the invariant set $A^c$ respectively and are defined similar to the restriction of the P-F operator in the autonomous case in equation (2). Stabilization of invariant set is posed as a co-design problem of jointly obtaining the control Lyapunov measure and the control P-F operator $\mathbb{P}_C$ or in particular the conditional probability density function $f(a|x)$. The co-design problem is formulated as an infinite dimensional linear program after suitable change of coordinates. Computational method based on set oriented numerical approach is proposed for the finite dimensional approximation of linear program in [10]. The goal of this
paper and the following sections is to extend the Lyapunov measure based framework for the optimal stabilization of an attractor set. One of the main differences between the results in [10] and this paper is that the finite deterministic optimal control is obtained as a solution of finite linear program as opposed to mixed integer linear program in [10].

III. Optimal Stabilization

The objective is to stabilize the invariant set $A$ using feedback control input $u = K(x)$, while minimizing a relevant cost function. We assume that the invariant set is locally stabilized and hence forms an attractor set. In the context of the controlled system, the invariant set $A$ is stabilized for all initial conditions starting from the set $B \subset X_1 := X \setminus U_\delta$ is denoted by $\mathcal{C} (B)$ and is given by the following formula

$$\mathcal{C} (B) = \int_{B} \gamma^D G(x_n, u_n) dm(x)$$

where $x_0 = x$, $x_{n+1} = T(x_n, u_n)$ for $n \geq 0$. For a given stabilizing feedback controller mapping $C(x) = (x, K(x))$, the cost of stabilization $\mathcal{C} (B)$ is given by

$$\mathcal{C} (B) = \int_{B} \sum_{n=0}^{\infty} \gamma^D C(x_n) dm(x)$$

Note that the cost of global stabilization of the invariant set $A$ is the special case of $B = X$. For (6) to be finite, when $\gamma > 1$, we require that the controller $C(x)$ is not just stabilizing but stabilizing the invariant set $A$ with geometric decay rate $\beta < \frac{1}{2}$. In the following theorem we show that the cost of stabilization of the invariant set $A$ can be expressed using the control Lyapunov measure $\mathcal{L}_G$. We first introduce the Koopman operator which is to be used in the proof.

Definition 5 (Koopman operator): For a continuous mapping $F : X_1 \to X_2$, Koopman operator $U_F : \mathcal{C}^0 (X_2) \to \mathcal{C}^0 (X_1)$ is given by

$$(U_F f)(x) = f(F(x))$$

where $f \in \mathcal{C}^0 (X_2)$ and $\mathcal{C}^0 (X_1)$ is the space of all continuous functions on compact spaces $X_i$ for $i = 1, 2$.

The P-F and Koopman operator are dual to each other and the duality is expressed using the following inner product

$$\langle U_F f, \mu \rangle_{X_1} = \int_{X_1} f(F(x)) d\mu(x) = \int_{X_2} f(x) d\mu (\mu) = (f, \mu)_{X_2} (7)$$

The result on the cost of stabilization follows.

Theorem 6: Let $\gamma > 1$ in the cost function and the controller mapping $C(x)$ is designed such that the invariant set $A$ is a.e. stable with geometric decay rate $\beta < \frac{1}{2}$. The cost of stabilization of an invariant set $A$ w.r.t. Lebesgue measure initial conditions starting from set $B$ is given by

$$\mathcal{C} (B) = \int_{B} \sum_{n=0}^{\infty} \gamma^D G(x_n, u_n) dm(x)$$

where $\bar{\mu}_B$ is the solution of the following control Lyapunov measure equation

$$\gamma^D \mathcal{L}_G \cdot \mathcal{D} \bar{\mu}_B (D) - \bar{\mu}_B (D) = -m_D (D)$$

for every set $D \subset X_1$.

Proof: From the assumption of a.e. geometric stability of the controller mapping $C : X \to Y$, we have by Theorem 4 that there exists non-negative measure $\bar{\mu}$ which is finite on $\mathcal{D} (X_1)$ and satisfies

$$\gamma^D \mathcal{L}_G \cdot \mathcal{D} \bar{\mu} (D) - \bar{\mu} (D) = -m(D).$$

For the cost of stabilization of a set $B$, we have

$$\mathcal{C} (B) = \int_{B} \sum_{n=0}^{\infty} \gamma^D G \circ C(x_n) dm(x) = \lim_{N \to \infty} \int_{B} \sum_{n=0}^{N} \gamma^D G \circ C(x_n) dm(x)$$

where $f_N(x) = \sum_{n=0}^{N} \gamma^D G \circ C(x_n)$ and $x_0 = x$. Using monotone convergence theorem and $G \geq 0$, $f_N(x) \leq f_{N+1}(x)$, we have

$$\lim_{N \to \infty} \sum_{n=0}^{N} \gamma^D G \circ C(x_n) dm(x) = \lim_{N \to \infty} \sum_{n=0}^{N} \langle \gamma^D G \circ C(x_n), m_B \rangle_{A'}$$

where we have used the fact that $x_n = (T \circ C)^n(x)$ and the duality between the operators $\mathcal{L}_G$ and $\mathcal{L}_C$. Let $\bar{\mu}_B = \sum_{n=0}^{N} \gamma^D G \circ C(x_n) dm_B$. The measure $\bar{\mu}_B$ is absolutely continuous with respect to Lebesgue measure $m$ for all $N$ since for any set $D \subset X_1$ if $m(D) = 0$ then $(\mathcal{L}_G \cdot \mathcal{D} m_B) (D) = m((T \circ C)^{-n}(D) \cap B) = 0$ for all $n$ and every set $B \subset X_1$. The latter is true because of the non-singularity assumption of the closed loop map $T \circ C$. Moreover $\bar{\mu}_B (D) \leq m_B^{N+1}(D)$ for every set $D, B \subset X_1$. Hence there exists an integrable function $g_N(x) \geq 0$ such that $m_B (D) \leq g_N(x) dm_B(x)$. Hence we have

$$\lim_{N \to \infty} \langle \mathcal{L}_G \cdot \sum_{n=0}^{N} \gamma^D \mathcal{L}_C \cdot m_B \rangle_{A'} = \lim_{N \to \infty} \int_{A'} \langle \mathcal{L}_G \cdot \sum_{n=0}^{N} \gamma^D \mathcal{L}_C \cdot m_B \rangle_{A'} = \int_{A'} \mathcal{L}_G (x) dm(x)$$

where $\bar{\mu}_B := \sum_{n=0}^{\infty} \gamma^D \mathcal{L}_C \cdot m_B = \sum_{n=0}^{\infty} \gamma^D \mathcal{L}_C \cdot m_B$ and $\bar{\mu}_B$ is known to be finite on any set $D \subset X_1$ because of a.e. stability property of the invariant set $A$ with geometric decay rate $\beta < \frac{1}{2}$. Furthermore $\bar{\mu}_B$ satisfies following control Lyapunov measure equation (3). Finally using the duality between $\mathcal{L}_G$ and $\mathcal{L}_C$, we get

$$\langle \mathcal{L}_G, \bar{\mu}_B \rangle_{A'} = \langle G, \mathcal{D} \bar{\mu}_B \rangle_{A'}$$

which proves the claim.

The minimum cost of stabilization is defined as the minimum over all a.e. stabilizing controller mapping $C$ as follows:

$$\mathcal{C}^* (B) = \min_{C} \mathcal{C} (B)$$

Defining $\theta (O) := \mathcal{D} \mu (O)$ for any set $O \subset X_1 \times U$, $\theta \in \mathcal{M} (A \times U)$, $\mu \in \mathcal{M} (A')$, the inner product $(f, \mu)_{X} :=$
optimal linear program for stabilization as follows

\[ \min \{ G, \theta \}_{A' \times U} \]  

s.t. \[ \gamma [P_1^\theta](D) - [P_1^\theta](D) = -m_B(D) \ \forall D \subset X_1 \]  

Using these definition, we can pose the infinite dimensional optimal linear program for stabilization as follows

\[
\begin{align*}
\min_{\theta \geq 0} \langle G, \theta \rangle_{A' \times U} \\
\text{s.t.} \quad \gamma [P_1^\theta](D) - [P_1^\theta](D) = -m_B(D) \quad \forall D \subset X_1
\end{align*}
\]

In the next section, we propose a computational framework based on the set oriented numerical methods for the finite dimensional approximation of the optimal stabilization problem. Optimal control for stabilization is obtained using the finite dimensional approximation of the linear program (11).

IV. COMPUTATIONAL APPROACH

The objective of the present section is to present a computational framework for the solution of the infinite-dimensional description is replaced by its finite-dimensional approximations. We assume a finite partition of state-space \( X \), and denote it by \( \mathcal{Y} := \{Y_1, \ldots, Y_N\} \), together with the associated measure space \( \mathbb{R}^N \). We assume without loss of generality that the invariant set, \( A \) is contained in \( D_N \), that is \( A \subseteq D_N \). Typically, the size of \( D_N \) is determined by the region of stability of a local controller that attempts to stabilize the neighborhood of the invariant set. We also assume that for two partitions \( \mathcal{Y}_1 := \{D_1, \ldots, D_N\} \) and \( \mathcal{Y}_2 := \{E_1, \ldots, E_{N'}\} \) we have \( D_N = E_{N'} \). Similarly the control space \( U \) is quantized and the control input is assumed to take only finitely many control values from the quantized set

\[ \mathcal{U}_M = \{u^1, \ldots, u^M\} \]

In addition, we will utilize the concept of a sub-partition which we define below. A partition \( \mathcal{Y}_{N'} := \{E_1, \ldots, E_{N'}\} \) for some \( N' > N \) is called a sub-partition of \( \mathcal{Y}_N \) if for each \( E_j \) there exists unique \( j \) \( \in \{1, \ldots, N\} \) such that \( E_i \cap D_j \neq \emptyset \) and \( E_i \cap D_k = \emptyset \) for all \( k \neq j \).

The partition for the joint space \( Y \), denoted by \( \mathcal{J} \), has cardinality \( N \times M \) and is identified with an associated vector space \( \mathbb{R}^{N \times M} \). We use the notation \( P_{T_a} \in \mathbb{R}^{N \times N} \) to denote the finite dimensional counterpart of P-F operator \( P_T \) resulting from fixing the control action on the state-space to \( u^a \), that is \( u(D) = u^a \) for all \( D \subset \mathcal{Y}_N \). The entries of \( P_{T_a} \) are calculated as:

\[
(P_{T_a})_{ij} := \frac{m(T_{a}^{-1}(D_i) \cap D_j)}{m(D_j)}
\]

where \( m \) is the Lebesgue measure and \( T_{a}^{-1} := T(\cdot, u^a) \). Since \( T : Y \rightarrow X \), we have that \( P_{T_a} \) is a Markov matrix. Additionally, \( P_{T_a} : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^{N-1} \) will denote the finite dimensional counterpart of P-F operator restricted to the complement of the invariant set, \( \mathbb{P}_T \) with the control input is fixed to \( u^a \). It is easily seen that \( P_{T_a} \) consists of the first \( (N-1) \) rows and columns of \( P_{T_a} \).

With the above quantization of the control space and partition of the state space, the determination of the control \( u(x) \in U \) (equivalently \( K(x) \)) for all \( x \in A' \) has now been cast as a problem of choosing \( u(D_i) \in \mathcal{U}_M \) for all sets \( D_i \in \mathcal{Y}_N \). The finite dimensional approximation of the minimum cost of stabilization (11) is equivalent to solving the following finite-dimensional linear program:

\[
\begin{align*}
\min_{\theta^a, \mu \geq 0} \sum_{a=1}^{M} (G^a)' \theta^a \\
\text{s.t.} \quad \gamma \sum_{a=1}^{M} (P_{T_a})^{' \theta^a} - \mu = -m \\
\sum_{a=1}^{M} \theta^a = \mu
\end{align*}
\]

where we have used the notation \((\cdot)'\) for the transpose, \( m \in \mathbb{R}^{N-1} \) and \( m_j \geq 0 \) denotes the discrete counterpart of the Lebesgue measure \( m(B) \) in (11), \( G^a \in \mathbb{R}^{N-1} \) and \( G^a_j \) is the cost associated with using control action \( u^a \) on set \( D_j \), \( \theta^a, \mu \in \mathbb{R}^{N-1} \) are respectively the discrete counter-parts of infinite-dimensional measure quantities in (11).

In the linear program (13) we have not enforced the constraint

\[ \theta^a > 0 \text{ for exactly one } a \in \{1, \ldots, M\} \]  

for each \( j = 1, \ldots, (N-1) \). The above constraint ensures that the control obtained is deterministic. We prove in the following that a deterministic controller can always be obtained provided the linear program (13) has a solution. To this end, we introduce the dual linear program (18) associated
with the linear program in (13). The Lagrangian dual to the linear program obtained on elimination of the variable \( \mu \) (by substituting the relation (13c) in (13b)) is,
\[
\max \min_V m'V + \min_{\theta^a \geq 0} \left\{ \sum_{a=1}^{M} (G^a \theta^a + V) \left( \gamma \sum_{a=1}^{M} (P_{E_a}) \theta^a - \sum_{a=1}^{M} \theta^a + m \right) \right\}.
\]
(15)
The Lagrangian dual has a finite solution only when,
\[
G^a + \gamma P_{E_a} V - V \geq 0 \quad \forall a = 1, \ldots, M.
\]
Hence, the dual to linear program in (13) is,
\[
\max_V m'V \quad \text{s.t.} \quad V \leq \gamma P_{E_a} V + G^a \quad \forall a = 1, \ldots, M.
\]
(16a)
(16b)
In the above linear program (16), \( V \) are the dual variables to the equality constraints (13b).

**B. Existence of stabilizing controls for a partition**

The first step in showing the existence of solutions to the finite linear program (13) is proving the existence of a feasible solution. In this section, we will show the feasibility of the finite linear program (13) under the assumption that the invariant set \( A \) is almost everywhere stable for the dynamical system \( T(\cdot, \cdot) \) using the finite number of control values, \( \mathcal{U}_M \).

**Assumption 7** (Existence of a stable fine-partition):

There exists a partition of the state-space \( \mathcal{X}_N := \{E_1, \ldots, E_{i_1}, \ldots, E_N\} \) with \( N' \) sufficiently large and associated controls \( u(E_i) \in \mathcal{U}_M \) such that the system is coarse stable.

**Remark 8:** Although we do not have a proof, intuition suggests that assumption 7 is likely to hold true and is necessary for the existence of the finite dimensional controller.

In the following we show stability of the partition \( \mathcal{X}_N \) using stability of its sub-partition \( \mathcal{X}_{N'} \). Key to this is the Markov chain representation of the dynamics of the finite-dimensional state-space system. One of the important concepts we will use is that of hitting time. We define hitting time as the smallest number of time-steps in which a transition from any set of the partition to the set containing invariant set is possible.

**Definition 9:** Given a partition \( \mathcal{X}_N := \{D_1, \ldots, D_N\} \) of the state-space \( X \) with invariant set \( A \subseteq D_N \) and a prescribed choice of control action \( u_N(D_i) \) on each set of the partition, the hitting time \( \tau(D_i; (\mathcal{X}_N, u_N)) \) of a set \( D_i \) is defined as,
\[
\tau(D_i; (\mathcal{X}_N, u_N)) := \min \left\{ n \left| (P_{E_1} u_N)_i (P_{E_2} u_N)_{i_1} \cdots (P_{E_{i-2}} u_N)_{i_{n-1}} (P_{E_{i-1}} u_N)_{i_{n-1}N} > 0 \right. \right. \text{for some } i_1, \ldots, i_{n-1} \in 1, \ldots, (N-1) \right\}
\]
(17)
where \( P_{E_1} \in \mathbb{R}^{N' \times N'} \) denotes the finite dimensional approximation of the P-F operator.

We state the following result on the coarse stability of a partition using the definition above.

**Lemma 10:** Let \( P_{E_1} \) be the closed loop sub-Markov matrix constructed on the partition \( \mathcal{X}_N \) using the finite dimension control \( u_N \). The sub-Markov matrix \( P_{E_1} \) is transient if and only if
\[
\tau(D_i; (\mathcal{X}_N, u_N)) < N \quad \forall D_i \in \mathcal{X}_N.
\]

**Proof:** Consider the only if part first. Let the partition \( \mathcal{X}_N \) be coarse stable with choice of controls \( u_N(D_i) \). We first show that the hitting time is finite for all the sets in the partition. Suppose, there exists \( D_i \in \mathcal{X}_N \) such that \( \tau(D_i; (\mathcal{X}_N, u_N)) = \infty \). Consequently, there exists a partition of the index set \( \{1, \ldots, (N-1)\} = I_1 \cup I_2 \) where \( I_1 \) and \( I_2 \) are disjoint and for each \( j \in I_1 \) the hitting time of the set \( D_j \) is finite and for each \( j \in I_2 \) the hitting time of the set \( D_j \) is infinite (in particular, \( i \in I_2 \)). The restriction of the finite dimensional P-F approximation to the complement of the invariant set \( P_{E_1} \) can then be shown to have following structure,
\[
P_{E_1} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}
\]
where \( P_1 \in \mathbb{R}^{i_1 \times i_1} \) and \( P_2 \in \mathbb{R}^{(i_2 \times i_2)} \). Further, since the hitting time is infinite for all \( j \in I_2 \) we have that all row sums of the matrix \( P_2 \) is 1. By the Perron-Frobenius Theorem for non-negative matrices [19] this implies that \( \rho(P_2) = 1 \). From the block-diagonal structure we have that \( \rho(P_{E_1}) = 1 \) which contradicts the assumption that \( P_{E_1} \) is transient. Hence, we have that the hitting time for all sets is finite. Now we show that the hitting time is less than \( N \). For some set \( D_i \), let \( \tau(D_i; (\mathcal{X}_N, u_N)) = n \) and the set of indices be \( i_1, \ldots, i_{n-1} \) such that \( P_{E_1} u_N i_{n-1}N > 0 \). Suppose \( i_k = i_k \) for some \( 1 \leq l < k \leq n-1 \) then, eliminating indices \( i_{k+1}, \ldots, i_k \) we obtain a shorter sequence which satisfies \( P_{E_1} \). This contradicts the minimality in the definition of the hitting time (17). Thus for each set \( D_i \) the indices in sequence \( i_1, \ldots, i_{n-1} \) must all be distinct and hence, \( \tau(D_i; (\mathcal{X}_N, u_N)) < N \).

Consider the if part of the claim. Let, \( L := \max \tau(D_i; (\mathcal{X}_N, u_N)) \). Now, for any initial distribution supported on the complement of the invariant set, i.e. \( \mu \in \mathbb{R}^{N-1}, \mu \geq 0, \mu \neq 0 \), the evolution of the distribution after \( L \) transitions is given by \( \mu^L(P_{E_1}) \). From the definition of hitting time (17) we have that for any initial distribution on the complement of the invariant set there is a non-zero probability of entering the invariant set after \( L \) transitions. Hence, we have that
\[
\sum_{i=1}^{N-1} \mu^l(P_{E_1}) < \sum_{i=1}^{N-1} \mu_i.
\]
Since the above holds true for all initial distribution we must have that \( (P_{E_1})^m \to 0 \) as \( n \to \infty \). Hence, the sub-Markov matrix \( P_{E_1} \) is transient which implies coarse stability of the partition \( \mathcal{X}_N \) with the choice of controls \( u_N(D_i) \).

We will now prove the main result on the existence of stabilizing controls for a partition \( \mathcal{X}_N \) under Assumption 7. **Theorem 11:** Suppose Assumption 7 holds. Then, there exists stabilizing controls for any partition \( \mathcal{X}_N := \{D_1, \ldots, D_N\} \) of the state-space.

**Proof:** From Assumption 7, we have that there exists a fine-enough partition \( \mathcal{X}_N := \{E_1, \ldots, E_{N'}\} \) that is stabilizable
using the finite controls in \( \mathcal{U}_M \). In particular we assume that \( \mathcal{X}_N' \) is a sub-partition of \( \mathcal{X}_N \). Denote by \( u_N(E_i) \) the stabilizing controls on each of the sets \( E_i \in \mathcal{X}_N' \). Since, \( \mathcal{X}_N' \) is a sub-partition of \( \mathcal{X}_N \) we have that there exist nonempty sets \( S_i \subset \{1, \ldots, N'\} \) for \( i = 1, \ldots, N \) such that \( D_i = \bigcup_{j \in S_i} E_j \). Further, \( S_i \cap S_k = \emptyset \) for all \( i \neq k \).

Consider the following procedure for identifying the controls on sets of the partition \( \mathcal{X}_N \).

1. For each \( i = 1, \ldots, N \), let \( b_i := \arg\min_{j \in S_i} \tau(E_j; (\mathcal{X}_N', u_N)) \).
2. Let \( u_N(D_i) := u_N(E_{b_i}) \) for all \( i = 1, \ldots, N \).

We will show in the following that \( 0 \leq \tau(D_i; (\mathcal{X}_N', u_N)) < N \) for all \( i = 1, \ldots, N \) which by Lemma 10 ensures transient nature of \( P^1_{T_a} \) constructed over the partition \( \mathcal{X}_N \).

Prior to proving this, we will need some additional results relating the P-F matrices of the partitions. Let, \( P_{Ta} \) and \( Q_{Ta} \) represent respectively the P-F matrices corresponding to the partition \( \mathcal{X}_N \) and \( \mathcal{X}_N' \) for a fixed control action \( u^a \) on all sets of the partitions. We show that the P-F matrices are related.

By definition,
\[
(\mathcal{Q}_{Ta})_{ij} = \frac{m(E_i \cap T_a^{-1}(E_j))}{m(E_i)} \quad i,j = 1, \ldots, N'
\]
where \( T_a(\cdot) := T(\cdot, u^a) \). The entries of P-F matrix \( P_{Ta} \) are defined as,
\[
(P_{Ta})_{ij} := \frac{m(D_i \cap T_a^{-1}(D_j))}{m(D_i)} = \frac{m(D_i \cap T_a^{-1}(D_j \cap E_i))}{m(D_i)} = \frac{m(D_i \cap (\cup_{j \in S_i} E_j) \cap T_a^{-1}(D_j \cap E_i))}{m(D_i)}
\]
Since \( E_i \) are pair-wise disjoint we have that,
\[
T_a^{-1}(D_j \cap \cup_{i \in S_j} E_i) = \{x \in X | T_a(x) \in (D_j \cap \cup_{i \in S_j} E_i) \} = \{x \in X | T_a(x) \in (D_j \cap E_i) \} \cap \cup_{i \in S_j} \{x \in X | T_a(x) \in (D_j \cap E_i) \} = \cup_{i \in S_j} T_a^{-1}(D_j \cap E_i).
\]
Additionally, from the definition of a sub-partition we also have for a given \( j \) the sets \( T_a^{-1}(D_j \cap E_i) \) are pair-wise disjoint. Hence,
\[
(P_{Ta})_{ij} = \sum_{k \in S_j} \frac{m(D_k \cap T_a^{-1}(D_j \cap E_i))}{m(D_k)} = \sum_{k \in S_j} \frac{m(E_k \cap T_a^{-1}(E_j))}{m(D_k)} \quad (19)
\]
We are now ready to prove the claim of the theorem. Suppose that the choice of controls \( u_N \) is not stabilizing for the partition \( \mathcal{X}_N \). Then, there exists \( D_i \in \mathcal{X}_N \) such that \( \tau(D_i; (\mathcal{X}_N', u_N)) = \infty \). From the choice of controls we have that \( u_N(D_i) = u_N(E_{b_i}) \). Denote by \( a_i \) the index of control in \( \mathcal{U}_M \) corresponding to \( u_N(E_{b_i}) \), that is \( u_N(E_{b_i}) = u^{a_i} \in \mathcal{U}_M \). From the stability of the sub-partition \( \mathcal{X}_N' \), we know that \( \tau(E_{b_i}; (\mathcal{X}_N', u_N)) < N' \) and additionally we also have that there exists a non-empty set \( K \subseteq \{1, \ldots, N'\} \) such that \( \tau(E_k; (\mathcal{X}_N', u_N)) = \tau(E_{b_i}; (\mathcal{X}_N', u_N)) - 1 \) and \( (Q_{Ta})_{bk} > 0 \) \( \forall k \in K \).

Now we first eliminate the possibility that \( E_k \subset D_j \) for all \( k \in K \). This is not possible since it contradicts our choice of \( b_i \) in Step 2 for choosing the controls on each set of the partition. Hence, there exists \( k \in K \) such that \( E_k \subset D_j \) for some \( j \) different from \( i \). Since \( (Q_{Ta})_{bk} > 0 \), we have from (19) that \( (P_{Ta})_{ij} > 0 \). As a result, we have two possibilities:

- \( \tau(D_j; (\mathcal{X}_N', u_N)) < \infty \). If this is the case, then our assumption on the hitting time of set \( D_j \), \( \tau(D_j; (\mathcal{X}_N', u_N)) = \infty \) is contradicted and our claim is proved.
- \( \tau(D_j; (\mathcal{X}_N', u_N)) = \infty \). We can repeat the argument we made for \( D_i \) for the set \( D_j \). Note that \( b_j = b_i - 1 \). Given the finiteness of \( N \) and \( N' \) the above logic will terminate with \( b_j = 0 \) which implies that a set \( D_j \) with a non-zero one-step probability of transition to \( D_N \) has been identified and the claim is proved.

C. Existence of solutions to finite linear program

In this section, we show that existence of optimal deterministic controls solutions to the finite linear program (13). For the sake of simplicity and clarity of presentation, we will assume that \( m > 0 \) in the following. The results presented here can easily be extended to the case where \( m \geq 0, m \neq 0 \). We will first derive conditions under which the linear program (13) is feasible.

Lemma 12: Suppose that Assumption (7) holds and \( m > 0 \). Then, for any partition \( \mathcal{X}_N \) there exists \( \gamma > 1 \) such that for all \( \gamma \in [1, \gamma] \) there exists a feasible solution to the linear program (13).

Proof: From Theorem 11, we have that there exist stabilizing controls for the partition \( \mathcal{X}_N \). Hence, there exists a choice of controls \( u_N(D_i) \) on each cell of the partition such that the P-F matrix for the resulting system denoted by \( P_{Ta} \in \mathbb{R}^{N \times N} \) satisfies \( \rho(P_{Ta}) < 1 \). Hence, there exists \( \gamma > 1 \) such that \( \rho(\gamma P_{Ta}) = 1/\gamma \). Define,
\[
\theta^a_i := \begin{cases} x_i & \text{if } a_i = u_N(D_i) \forall i = 1, \ldots, (N-1) \\ 0 & \text{otherwise.} \end{cases}
\]
Note that with the above definition, the constraints of linear program (13) can be recast as
\[
(I_{N-1} - \gamma(\gamma^N P_{Ta})) x = m
\]
where \( I_{N-1} \in \mathbb{R}^{(N-1) \times (N-1)} \) is the identity matrix. Clearly, \( \rho(\gamma^N P_{Ta}) < 1 \) for all \( \gamma \in [1, \gamma] \) and as a result the following holds.
\[
x = (I_{N-1} - \gamma(\gamma^N P_{Ta}))^{-1} m = \sum_{n=0}^{\infty} ((\gamma P_{Ta})^n)m > 0.
\]
This proves the claim.

Remark 13: Lemma 12 has used the stabilizing controls identified in Theorem 11 to prove the existence of \( \gamma \). This choice of \( \gamma \) might be conservative and in fact, there may be other choices of controls which allow feasibility of (13) for larger values of \( \gamma \).

In the following we will derive conditions under which a solution to linear program (13) exists and then, show that
condition for deterministic control (14) can be satisfied under the assumption of feasibility of linear program (13). The main result is stated in Theorem 17.

**Lemma 14:** Suppose the Assumption 7 holds and $m > 0$, $G(\cdot, \cdot) \geq 0$ on the complement of the invariant set. Then for all $\gamma \in [1, \check{\gamma}]$, there exists an optimal solution $\theta$ to linear program (13) and an optimal solution $V$ to the dual linear program (16) with equal objective values ($\sum_{a=1}^{M} (G^\alpha) \theta^a = m'V$).

**Proof:** Assumption 7 ensures that the linear program (13) is feasible (Lemma 12). Observe that the linear program in (16) is always feasible with a choice of $V = 0$. The claims hold as a result of linear programming with strong duality [18].

**Remark 15:** Note that existence of an optimal solution does not impose positivity requirement on the cost function $G$ on the complement set. In fact, even assigning $G(\cdot, \cdot) = 0$ allows obtaining a stabilizing control from the Lyapunov measure equation (13b). In this case, any feasible solution to (13b)-(13c) is an optimal solution and Theorem 11 guarantees the existence of such a solution.

The next result shows that linear program (13) always admits a deterministic control action as an optimal solution. In the following, we will assume that the cost is positive on the complement of the invariant set $G(\cdot, \cdot) > 0$. This assumption is crucial in order to obtain deterministic controls.

**Lemma 16:** Suppose Assumption 7 holds and $m > 0$, $G(\cdot, \cdot) > 0$ on the complement of the invariant set. Let $\theta$ solve (13) and $V$ solve (16) for some $\gamma \in [1, \check{\gamma}]$. Then the following hold at the solution:

1) For each $j = 1, \ldots, (N-1)$ there exists at least one $a_j \in 1, \ldots, M$ such that $V_j = \gamma(\bar{P}_{T_a}) V_j + G^{\alpha}_j$ and $\theta^{a_j}_j > 0$ where $G^{\alpha}_j := G(D_i, u^{\alpha})$

2) There exists a $\tilde{\theta}$ that solves (13) and is such that for each $j = 1, \ldots, (N-1)$, there is exactly one $a_j \in 1, \ldots, M$ such that $\tilde{\theta}^{a_j}_j > 0$ and $\tilde{\theta}^{a}_j = 0$ for $a' \neq a_j$.

**Proof:** From the assumptions, we have that Lemma 14 holds. Hence, there exists $(V, \theta)$ that satisfy the first-order optimality conditions [18],

$$\sum_{a=1}^{M} \theta^a - \gamma \sum_{j=1}^{M} (\bar{P}_{T_a}) \theta^a = m$$

(20)

We will prove each of the claims in order.

1) Suppose, there exists $j \in 1, \ldots, (N-1)$ such that $\theta^a_j = 0$ for all $a = 1, \ldots, M$. Substituting in the optimality conditions (20) one obtains,

$$\gamma \sum_{j=1}^{M} (\bar{P}_{T_a}) \theta^a_j = -m_j$$

which cannot hold since, $\bar{P}_{T_a}$ has non-negative entries, $\gamma > 0$ and $\theta^a \geq 0$. Hence, there exists at least one $a_j$ such that $\tilde{\theta}^{a_j}_j > 0$. The complementarity condition in (20) then requires that $V_j = \gamma(\bar{P}_{T_a}) V_j + G^{\alpha}_j$. This proves the first claim.

2) Denote $a(j) = \min \{a | \theta_j^a > 0 \}$ for each $j = 1, \ldots, (N-1)$. The existence of $a(j)$ for each $j$ follows from statement 1. Define $\bar{P}_{T_a} \in \mathbb{R}^{(N-1) \times (N-1)}$ and $G^\alpha \in \mathbb{R}^{N-1}$ as follows:

$$\begin{align*}
(\bar{P}_{T_a})_{ji} := (\bar{P}_{T_{a(j)}})_{ji} & \forall i = 1, \ldots, (N-1) \\
G^\alpha_j := G^{\alpha(j)}_j
\end{align*}$$

(21)

for all $j = 1, \ldots, (N-1)$. From the definition of $\bar{P}_{T_a}$ and $G^\alpha$ and complementarity condition in (20) it is easily seen that $V$ satisfies,

$$V = \gamma\bar{P}_{T_a} V + G^\alpha = \lim_{n \to \infty} ((\gamma\bar{P}_{T_a})^n V + \sum_{k=0}^{n} (\gamma\bar{P}_{T_a})^k G^\alpha).$$

(22)

Since $V$ is bounded and $G^\alpha > 0$ it follows that $\rho(\bar{P}_{T_a}) < 1/\gamma$.

Define $\tilde{\theta}$ as follows,

$$\begin{bmatrix}
\tilde{\theta}^{a(1)}_1 \\
\vdots \\
\tilde{\theta}^{a(N-1)}_{N-1}
\end{bmatrix} = ((I_{N-1} - \gamma(\bar{P}_{T_a}))^{-1} m$$

(23a)

$$\tilde{\theta}^j_0 = 0 \forall j = 1, \ldots, (N-1), \ a \neq a(j).$$

(23b)

The above is well-defined since we have already shown that $\rho(\bar{P}_{T_a}) < 1/\gamma$.

From the construction of $\tilde{\theta}$, we have that for each $j$ there exists only one $a_j$, namely $a(j)$, for which $\tilde{\theta}^{a(j)}_j > 0$. It remains to show that $\tilde{\theta}$ solves (13). For this observe that,

$$\sum_{a=1}^{M} (G^\alpha) \tilde{\theta}^a = \sum_{j=1}^{N-1} G^{a(j)} \tilde{\theta}^{a(j)}_j = \sum_{j=1}^{N-1} G^{a(j)} ((I_{N-1} - \gamma(\bar{P}_{T_a}))^{-1} m$$

(24)

$$= ((I_{N-1} - \gamma(\bar{P}_{T_a}))^{-1} G^\alpha) m$$

$$= V^m.$$ 

The primal and dual objectives are equal with above definition of $\tilde{\theta}$ and hence, $\tilde{\theta}$ solves (13). The claim is proved.

**Lemma 16** shows that if there exists a solution to linear program (13) then, the there exists a deterministic controller for the same. The following theorem states the main result.

**Theorem 17:** Given the system $T : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ and $m > 0$, $G(\cdot, \cdot) > 0$ on the complement of the invariant set. If Assumption 7 holds then, the following statements hold for all $\gamma \in [1, \check{\gamma})$:

1) there exists a $\theta$ which is a solution to (13) and a $V$ which is a solution to (16)

2) the optimal control for each set $i = 1, \ldots, (N-1)$ is given by,

$$u(D_j) = \mu(a(j))$$

(25)

where $a(j) := \min \{a | \theta^a_j > 0 \}$

3) $\mu$ satisfying

$$\gamma(\bar{P}_{T_a}) \mu - \mu = -m \text{ where } (\bar{P}_{T_a})_{ji} = (\bar{P}_{T_{a(j)}})_{ji}.$$

(26)
is the Lyapunov measure for the controlled system.

Proof: Assumption 7 ensures that the linear programs (13) and (16) have a finite optimal solution (Lemma (14)). This proves the first claim of the theorem and also allows the applicability of Lemma 16. The remaining claims follow as a consequence.

Remark 18: In [10], we proposed solving mixed integer linear program to obtain deterministic control solution to the feedback stabilization problem using control Lyapunov measure. However using the results of Theorem 17, one can solve a linear program, with cost function $G = constant > 0$, to obtain deterministic feedback stabilizing control.

Though the results in this section have assumed that $m > 0$, this can be easily relaxed to $m \geq 0, m \neq 0$. The case of $m \geq 0, m \neq 0$ is of interest when the system is not everywhere stabilizable. If it is known that there are regions of the state-space that are not stabilizable, then the $m$ can be chosen such that its support is zero on those regions. If the regions are not known a priori then, the (13) can be modified to minimize the $l1$-norm of the constraint residuals. This is similar to the feasibility phase that is commonly employed in linear programming algorithms [20]. These and other ideas on computational complexity management will be addressed in a subsequent paper.

V. Example

We present the simulation result for optimal stabilization of period two orbit in quadratic Logistic map. The controlled Logistic map is described by the following equation.

$$x_{n+1} = ax_n(1-x_n) + u_n$$

where $x_n \in [0,1]$ is the state, $u_n$ is the control and the parameter $a = 4$. Figure (1a) shows the invariant measure for the uncontrolled Logistic map for the parameter value $a = 4$. Invariant measure gives us the steady state distribution of the points in the phase space. Invariant measure shows chaotic behavior with no stable periodic orbit of any period. Our goal is to stabilize a period two orbit. For Logistic map one can derive an analytical expression for the period-2 orbit in terms of the parameter $a$. The period-2 orbit points are given by the expression

$$x_{01} = \frac{a + 1 - \sqrt{a^2 - 2a - 3}}{2a}, \quad x_{02} = \frac{a + 1 + \sqrt{a^2 - 2a - 3}}{2a}$$

Hence for the parameter value $a = 4$ we get $x_{01} = 0.3455$ and $x_{02} = 0.9045$ as the unstable period-2 orbit. Figure (1) shows the simulation result for the stabilization of periodic two orbit for the Logistic map. The stabilization objective is achieved while minimizing the control input i.e., the cost function $G = x^2 + u^2$. For the finite dimension approximation we divide the interval $[0,1]$ into 300 equal length intervals. Similarly the control values ranges from $-0.05$ to $0.05$ in the steps of $0.01$.

Figure (1b) shows the plot of closed loop invariant measure. Figure (1c) shows the plot of control Lyapunov measure. Figure (1d) shows the control values used for stabilization. From this plot we see control is used only at discrete set of points thus exploiting the natural dynamics of the system.

The presence of eigenvalues at 1 and $-1$ for the closed loop system in figure (1e) implies the existence of stable period two orbit.

VI. Conclusions

The problem of optimal stabilization for discrete time nonlinear system is solved using linear transfer operator and Lyapunov measure based framework. Duality between Perron-Frobenius and Koopman operators is used to pose the primal and dual optimal stabilization problem as a infinite dimensional linear program. Computational framework based on set oriented numerical methods is used for the finite dimensional approximation of the optimal stabilization problem. This finite dimensional approximation of the optimal stabilization problem lead to solving finite number of linear inequalities. The highlight of the solution approach for the finite dimensional linear program is that the controller obtained is deterministic although the approximation of the linear transfer operators is stochastic. Simulation results for the optimal stabilization of period two orbit is presented. One of the main bottlenecks in the approach is that the size of the linear program scales as a function of the state-space discretization. Clearly, this becomes a huge problem for
higher dimensional systems. The solution of higher dimensional systems will require development of algorithms that exploit the structure of the problem. These and other ideas on computational complexity management will be addressed in a subsequent paper.

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