Markov Chains, Entropy, and Fundamental Limitations in Nonlinear Stabilization

Prashant G. Mehta  Umesh Vaidya  Andrzej Banaszuk

Abstract

In this paper, we propose a novel methodology for establishing fundamental limitations in nonlinear stabilization. To aid the analysis, we express the stabilization problem as control of Markov chains. Using Markov chains, we derive the limitations as certain maximum probability bounds or as positive conditional entropy of the certain signals in the feedback loop. The former is related to the infeasibility of the asymptotic stabilization in the presence of quantization and the latter to the Bode integral formula. In either cases, it is shown that uncertainty – associated here with the unstable eigenvalues of the linearization – leads to fundamental limitations.

I. INTRODUCTION

Recently, there have been several important works relating fundamental limitations in control (Bode formula) to entropy rates:

\[
\mathcal{H}_c(x) - \mathcal{H}_c(d) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |S(e^{i\omega})| d\omega = \log(a).
\]  

Here, \(S(e^{i\omega})\) is the sensitivity transfer function of the feedback loop from the disturbance \(d\) to the output \(x\), \(\mathcal{H}_c(\cdot)\) denotes the conditional entropy (see [1], [2]) of random process, and \(a\) is the expansion rate with \(\log(a) = \sum_k \log(|p_k|)\), \(p_k\) being the unstable poles of the open loop plant (\(|p_k| > 1\)). In [3], Martins and Dahleh employ methods of information theory to derive the entropy bound for the linear disturbance rejection problem. In [4], Zang and Iglesias obtain entropy estimates for the disturbance rejection problem where the open-loop plant is globally asymptotically stable and furthermore has a finite or fading memory property. In [5], it is shown that the rate of instability of the open-loop plant must be compensated by the information transmission rate over the communication channel in any stabilizing feedback.

Entropy is also relevant to the study of deterministic and stochastic nonlinear dynamical systems via methods of Ergodic theory; cf., [6]. One important notion is the topological entropy, that is used to estimate the growth rate of the number of distinguishable orbits on taking finer and finer partitions of the phase space. In [7], Nair et. al. extend this notion to feedback systems by defining topological feedback entropy (TFE). The authors express fundamental limitation results in nonlinear stabilization as bounds on TFE. These results, taken together with results of [3], [4], [5], have laid a foundation for expressing limitations in control of nonlinear plants with a well-defined linearization.

P. G. Mehta is with the Department of Mechanical Science & Engineering and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1206 W. Green Street, Urbana, IL 61801 mehtapg@uiuc.edu

U. Vaidya is with the Department of Electrical & Computer Engineering, Iowa State University, Ames, IA 50014 ugvaidya@iastate.edu

A. Banaszuk is with the Systems Department, United Technologies Research Center, East Hartford, CT 06108 banasza@utrc.utc.com
In this paper, we revisit the study of fundamental limitations by developing an explicit probabilistic framework for the study of deterministic control problems. The proposed methodological framework is based upon methods of ergodic theory whereby the deterministic nonlinear dynamical system is replaced by its stochastic counterpart, the so-called Perron-Frobenius operator [8]. While, the dynamical model propagates the initial condition, the Perron-Frobenius (P-F) operator propagates uncertainty in initial condition. There are three advantages to doing so. One, it is now easier to compute stochastic quantities such as entropy relevant to the study of fundamental limitations. Two, the P-F operator is linear enabling analysis in very general settings. Finally, the stabilization problem with finite partition, or quantization, is easily considered using the discretization of the P-F operator. The novel PF-based framework represents the main contribution of our work.

This paper is the first to consider measure-theoretic notions of entropy and bridging the statistical entropy notion of [4], [5], [3] with the Dynamical systems notion of the entropy [7]. The measure theoretic notions appear to be particularly well-suited for uncertainty characterization with disturbance whereby the nonlinear stabilization is interpreted as a suitable limit. Our work also establishes rigorous connections to Markov chains thereby creating an opportunity to bridge performance limitations in control with performance limitations in other literature, such as queueing networks, where Markovian models is the norm.

Discretization of the P-F operator also allow one to establish connections to the literature on control with quantization. To do so, we introduce the probabilistic notion of $q$-stability. Although, it is well known that asymptotic stabilization of a linearly unstable equilibrium is infeasible in the presence of discrete quantized feedback (see [9]), $q$-stability provides for a more refined characterization of this result. Our main result, expressed in Theorem 6, shows that the quantized interval can at best be made stable with probability $\frac{1}{a}$ where $a > 1$ is the expansion rate. This bound is also shown to be control independent and thus fundamental. Finally, we apply the proposed framework to re-derive Bode type entropy estimates for the problem of nonlinear stabilization. Our results, expressed in Theorems 8 and 11, are the same as (and motivated by) results of Nair et. al. [7] (although the authors there used topological notions of entropy) and the results pertaining to stabilization of Martins and Dahle [3] for the linear case. Also as in both these works, the estimates are control independent and in fact, quite general with respect to the nature of the stabilizing control used (linear, nonlinear, quantized or even certain probabilistic controls).

The outline of this paper is as follows. In Section II, we begin by describing the Perron-Frobenius formalism for a nonlinear dynamical system. In Section III, we formulate the nonlinear stabilization problem and in Section IV, we present the probability and entropy bounds for this problem.

II. PRELIMINARIES

In this paper, discrete mappings of the form

$$x_{n+1} = T(x_n)$$  \hspace{1cm} (2)

are considered. $T : X \to Y \subset \mathbb{R}^m$ is assumed to be continuous and $X, Y \subset \mathbb{R}^m$ are compact sets; often $X = Y$. $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra on $X$ and $\mathcal{M}(X)$ the vector space of bounded real valued measures on $\mathcal{B}(X)$. The Perron-Frobenius (P-F) operator for (2) is given by

$$\mathbb{P}[\mu](A) = \int_X \delta_{T_x}(A) d\mu(x) = \mu(T^{-1}(A)), \hspace{1cm} (3)$$

where $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$ and $\delta_{T_x}(A)$ is the stochastic transition function for the deterministic map (2); cf., [8]. By taking finite partitions of the compact sets $X$ and $Y$, one can approximate the infinite-dimensional...
P-F operator by a finite-dimensional matrix:

\[ P_{ij} = \frac{m(T^{-1}(E_j) \cap D_i)}{m(D_i)}, \]  

where \( \cup_i D_i = X, \cup_j E_j = Y \) and \( m \) is the Lebesgue measure; cf., [10]. This matrix is Markov (row stochastic) for stochastic \( \mathbb{P} \) and sub-Markov (row sum less than equal to 1) for sub-stochastic \( \mathbb{P} \). By defining an appropriate stochastic transition function (integrand in (3)), these considerations also extend to a class of randomly perturbed dynamical systems

\[ x_{n+1} = T(x_n, d_n), \]  

where \( d_n \in D \) models a disturbance assumed in this paper to be i.i.d with probability measure \( \Omega(D) \); cf., [11]. Notationally, \( P \) is used to represent a finite-dimensional Markov matrix and \( \mathbb{P} \) a continuous P-F operator. We will use the underline notation to distinguish a finite-dimensional measure \( \mu \in \mathbb{R}^L \) (a vector) from an infinite-dimensional \( \mu \in \mathcal{M}(X) \); the \( i \)th element of the finite-dimensional vector \( \mu \) is denoted as \( \mu_i \). We refer the reader to our paper [12] (and references therein) for additional details on the PF formalism.

The stabilization problem is considered for the closed-loop equation

\[ x_{n+1} = T \circ (1 + K)(x_n) \equiv T_K(x_n), \]  

where \( T \) is the plant and \( 1 + K \) is the control; \( 1 \) denotes the identity mapping and \((1 + K)(x_n) \equiv x_n + K(x_n)\). It is assumed that \( T(0) = 0 \) and that \((1 + K)(0) = 0 \). The control is said to be inactive if \( K = 0 \). The control \( K \) is said to be stabilizing if the linearization of \( T_K \) taken at the equilibrium 0 has eigenvalues strictly inside the unit circle. For such a case, we note that the following holds for for sufficiently small \( \varepsilon \): There is an \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in [0, \varepsilon_0] \), there exists a compact set \( X^{\varepsilon} \) with \( 0 \in X^{\varepsilon} \) and volume \( m(X^{\varepsilon}) = \varepsilon \) (or \( O(\varepsilon) \)) such that \( T_K : X^{\varepsilon} \to X^{\varepsilon} \). This may be proved by using a local Lyapunov function in conjunction with the Grobman-Hartman theorem [6].

### III. CONTROL PROBLEM FORMULATION

For the sake of exposition in this section, we consider the closed-loop equation (6) on some compact set \( X \subset \mathbb{R}^n \) where \( 0 \in X \), where \( T \circ (1 + K) : X \to X \), the mapping \( T \) is assumed to be well-defined for all \( x \in X \), and furthermore it is assumed that \((1 + K) : X \to X \). The assumption \((1 + K) : X \to X \) is stronger than what is needed. However, the exposition is simpler with only a single compact set \( X \) and the scalar stabilization problem and certain multi-state problems can be handled with this formulation. In order to prove the multi-state stabilization result, we will extend it to handle that particular case.

In addition to Eq. (6), a randomized or perturbed version of the closed-loop equation

\[ x_{n+1} = T_d \circ (1 + K_d)(x_n), \]  

is also considered, where \( T_d \) and \( 1 + K_d \) are randomly perturbed maps. Often,

\[ T_d(u) \equiv T(u) + d \]  

where \( d \) is a random variable taking values from a given distribution. In fact, Eq. (7) then corresponds to a closed-loop with disturbance:

\[ x_{n+1} = T \circ [(1 + K)(x_n) + d^K_n] + d^T_{n+1}, \]  

where \( \{d^K_n\} \) and \( \{d^T_n\} \) are disturbance processes, each modeled here as an i.i.d random process.
**Example 1** Consider the closed-loop system with linear equations:

\[
\begin{align*}
\text{state: } z_{n+1} &= az_n + bu_n, \\
\text{output: } x_n &= z_n + d_n, \\
\text{control: } u_n &= \tilde{k}(x_n) + \tilde{r}_n.
\end{align*}
\] (10)

The closed-loop equation for the output is given by

\[
x_{n+1} = a((1+k)x_n + r_n - d_n) + d_{n+1},
\] (11)

where \(k = \frac{b}{a}\tilde{k}\), and \(r_n = \frac{b}{a}\tilde{r}_n\). These considerations also extend to the multi-state case. The perturbed closed-loop in Eq. (9) is thus the nonlinear generalization of the linear disturbance rejection problem with full-state feedback. With no disturbance, one recovers the nonlinear stabilization problem in Eq. (6) as the special case of Eq. (9).

Denote \(P_T\) and \(P_K\) to be the infinite-dimensional P-F operators corresponding to mappings \(T\) and \(1+K\), respectively. The propagation of measures (in \(\mathcal{M}(X)\)) for the composition in Eq. (6) is easily verified to be given by

\[
\mu_{n+1} = \mu_n P_K \cdot P_T,
\] (12)

where the ordering reflects the fact that controller \((P_K)\) acts before the plant \((P_T)\). This formalism is useful because 1) PF operators are linear and 2) nonlinear composition of two mappings lead to linear multiplication of corresponding operators. If control is inactive \((K = 0)\) then \(P_K = 1\), the identity operator. A similar stochastic description also exists for the composition in Eq. (7), where the P-F operators now correspond to the random mappings. Closely related to Eq. (12) is its finite-dimensional approximation which is discussed next.

A. Conditional Dither

In order to pose a stabilization problem with respect to a finite partition, we propose the use of *conditional dither* that is defined with respect to the partition. Given a partition with a quantization:

\[
\mathcal{X} = \{D_1, \ldots, D_L\},
\]
(13)
\[
\mathcal{Q} = \{\theta(1), \ldots, \theta(L)\},
\]
(14)

\(\theta(i) \in D_i\) being the quantized value in cell \(D_i\), conditional dither is defined using a random vector

\[
\mathcal{D} = \{\partial(1), \ldots, \partial(L)\},
\]
(15)

where \(\partial(i)\) represents a uniformly distributed random perturbation with support on cell \(D_i\). In particular, given \(z \in X\), define a random variable

\[
y(z) = \theta(|z|) + \partial(|z|),
\]
(16)

where \(|z| = i\), the subscript of the cell \(D_i\) where \(z\) lies. Intuitively, \(\partial(|z|)\) randomizes the state \(z\) within the cell. \(y\) and \(z\) are both located in the same cell \(D_i\) and contain the same information modulo the partition. We will refer to Eqs. (13)-(15) as a finite partition with conditional dither.

With a finite partition, the stabilization problem is posed with respect to the perturbed closed-loop equation

\[
x_{n+1} = T_d \circ (1 + K_d)(x_n),
\]
(17)
where the subscript $d$ corresponds to the conditional dither. In particular, we use $\mathcal{D}$ to define
\[
T_d(u) \coloneqq \theta(|T(u)|) + \partial(|T(u)|),
(1 + K_d)(u) \coloneqq \theta(|(1 + K)(u)|) + \partial(|(1 + K)(u)|).
\]
Formally, $T_d$ and $K_d$ approximates $T$ and $K$ in the limit of taking finer partitions. More precisely, $T_d$ corresponds to the stochastic transition function
\[
p_d(x,A) = \int_A \chi_{D_{T(x)}}(y) \frac{dm(y)}{m(D_{T(x)})},
\]
where $m$ is the Lebesgue measure representing the fact that disturbance is taken from a uniform distribution. $m(D_{T(x)})$ is used for normalization so $p_d(x,A)$ is a probability measure. The stochastic transition function corresponds to a small random perturbation of the deterministic dynamical system $T$, i.e., $p_d(x,\cdot) \to \delta_{T(x)}$ in a weak-* sense in the limit as $m(D_j) \to 0$ for all cells. For $T_d$, the Markov operator is given by
\[
\mathbb{P}^d[\mu](A) = \int_X p_d(x,A) d\mu(x)
\]
whose invariant measure converges to the physical measure of $T$ as $d \to 0$; cf., [13], [11]. The following lemma clarifies the relationship between stochastic aspects of $T_d$ and the Markov chain $P_T$.

**Lemma 2** Consider a dynamical system $T : X \to X$ with a finite partition $\mathcal{X}_L$ and an associated Markov chain $P_T$. Using conditional dither $\mathcal{D}$, consider next a randomly perturbed dynamical system $T_d : X \to X$ and the associated Markov operator $\mathbb{P}^d$.

1) Suppose $\mu \in \mathcal{M}(X)$ and $\nu = \mathbb{P}^d[\mu]$. Then there exists $\nu_i$ such that
\[
d\nu(x) = \sum_{i=1}^L \nu_i \chi_{D_i}(x) \frac{m(D_i)}{m(D_j)} dm(x).
\]

2) If $\nu_j = \sum_i \nu_i P_{ij}$ then $\nu = \mathbb{P}^d[\mu]$, where
\[
\frac{d\mu}{dm}(x) = \sum_{i=1}^L \nu_i \chi_{D_i}(x) \frac{m(D_i)}{m(D_j)}, \quad \frac{d\nu}{dm}(x) = \sum_{i=1}^L \nu_i \chi_{D_i}(x) \frac{m(D_i)}{m(D_j)}.
\]

3) Suppose $P_T$ admits a unique invariant measure $\mu = [\mu_1, \ldots, \mu_L]$. Then starting with Lebesgue a.e. set of initial conditions $x_0 \in X$, the trajectory $\{x_n\}$ of the map $T_d$ has a stationary distribution given by
\[
\text{Prob}(x_n \in D_i) = \mu_i,
\]
and it is uniformly distributed in $D_i$.

**Proof:** We note that $T_d : X \to X$ as $T : D_i \to X$ and the conditional dither perturbs only within the cells of the partition. For $\mu \in \mathcal{M}(X)$,
\[
v(A) = \mathbb{P}^d[\mu](A) = \int_X \int_A \chi_{D_{T(x)}}(y) \frac{dm(y)}{m(D_{T(x)})} d\mu(x).
\]
Now, $\frac{1}{m(D_{T(x)})} \chi_{D_{T(x)}}(y) = \sum_{j=1}^L \chi_{D_j}(y) \cdot \chi_{D_j}(Tx) \frac{1}{m(D_j)}$ and substituting this in Eq. (24), we have
\[
v(A) = \int_A \sum_{j=1}^L \int_X \chi_{D_j}(Tx) d\mu(x) \frac{\chi_{D_j}(y)}{m(D_j)} dm(y).
\]
Denoting \( v_j = \int_X \chi_{D_j}(Tx) \, d\mu(x) \), Eq. (21) and part (1) follows. In order to show part (2), use Eq. (22) to evaluate these coefficients explicitly:

\[
v_j = \int_X \chi_{D_j}(Tx) \sum_{i=1}^L \mu_i \chi_{D_i}(x) \, m(D_i) = \sum_{i=1}^L \mu_i \left( \int_X \chi_{D_i}(x) \, \chi_{D_j}(x) \, dm(x) \right) = \sum_{i=1}^L \mu_i P_{ij}.
\]

as desired. From part (2), if \( \mu P_T = \mu \) then \( \mathbb{P}^d[\mu] = \mu \), where \( dm(x) = \sum_{i=1}^L \mu_i \chi_{D_i}(x) \). Using part (1), if \( \mathbb{P}^d[\mu] = \mu \) then \( \mu \) too has this form and thus \( \mu P_T = \mu \). As a result, invariant measures of \( P_T \) are in one-one correspondence with invariant measures of \( \mathbb{P}^d \). Finally, we note that the stationary density of the random sequence \( \{x_n\} \) is given by

\[
f(x) = \frac{\mu_i}{m(D_i)} \quad \text{for } x \in D_i.
\]  

The following theorem, which follows from the Lemma 2, clarifies the relationship between the asymptotic behavior of the perturbed closed-loop Eq. (17) and the invariant measure of the discrete formulation \( P_K \cdot P_T \).

**Theorem 3** Consider a partition \( \mathcal{D}_L \) with disturbance \( \mathcal{D} \). Let \( P_T \) and \( P_K \) be the sub-Markov (or Markov) chain for \( T \) and \( (1 + K) \) respectively. Suppose \( P_K \cdot P_T \) is a Markov chain (with row sum as unity) then:

1. The perturbed closed-loop map \( T_\alpha \circ (1 + K_\alpha) : X \to X \),
2. If \( P_K \cdot P_T \) has a unique invariant measure \( \mu = [\mu_1, \ldots, \mu_L] \) then starting with (Lebesgue) a.e. initial conditions \( x_0 \in X \), the trajectory \( \{x_n\} \) for the closed-loop Eq. (17) has a stationary distribution given by

\[
\text{Prob}(x_n \in D_i) = \mu_i,
\]

and it is uniform in \( D_i \).

**Proof:** We prove the first part by contradiction. Suppose there is a set \( B \subset X \) of positive Lebesgue measure such that \( T \circ (1 + K_\alpha)(B) \not\subset X \) for some value of disturbance \( d \). Now, because there are only finitely many cells \( \{D_i\} \), there must at least be one cell \( D_i \) such that \( m(B \cap D_i) > 0 \). So without loss of generality, we assume \( B \subset D_i \) such that \( m(B) > 0 \) and \( T \circ (1 + K_\alpha)(B) \not\subset X \). Next because \( (1 + K) : X \to X \), let \( \{D_{jr}\}_{r=1}^R \) be the \( R \) cells such that \( (1 + K)(D_m) \cap D_{jr} \not= \{\phi\} \) and \( m((1 + K)(D_m) \cap D_{jr}) > 0 \). We denote \( D_q \triangleq D_{j1} \) and note that the last statement implies that

\[
[P_K]_{qj} > 0.
\]

We use the notation

\[
[P_T]_{q} = [p_1, \ldots, p_L]
\]

to denote the \( q \)-th row and note that because \( T \circ (1 + K_\alpha)(B) \not\subset X \) we have \( \sum_{i=1}^L p_i = \beta_q < 1 \). Eqs. (29)-(30) will be used to derive a contradiction by showing that the \( i \)-th row sum of \( P_K \cdot P_T \) is strictly less than 1. Indeed on multiplying \( P_T \) on the right by column vector \( 1 \), one obtains

\[
P_T \cdot 1 = [\beta_1, \ldots, \beta_q, \ldots, \beta_L],
\]

where \( \beta_i \leq 1 \) and \( \beta_q < 1 \). Next denote the \( i \)-th row \( [P_K]_i = [\alpha_1, \ldots, \alpha_q, \ldots, \alpha_L] \) and \( \alpha_q > 0 \) by Eq. (29). The \( i \)-th row sum is now easily seen to be

\[
\sum_{i=1}^L \alpha_i \beta_i \leq \max(\beta_i) \sum_{i \neq q}^L \alpha_i + \beta_q \alpha_q \leq 1 \cdot \sum_{i \neq q}^L \alpha_i + \beta_q \alpha_q < 1.
\]
As a result, $P_K \cdot P_T$ is not a Markov chain providing the contradiction we seek. This proves part (1).

The part (2) of the theorem follows from the final part of Lemma 2. Assume an invariant measure $\mu P_K \cdot P_T = \mu$ and denote $\mu P_K = \nu$ so $\nu P_T = \mu$. Using Lemma 2, $\mu P_K \cdot P_T = \nu$ so $\mu P_K = \mu$ where $\frac{d\mu}{dm}(x) = \sum_{i=1}^{L} \mu_D(x_i)$. The result then follows by using (assuming) a suitable Ergodic hypothesis for invariant measures.

The significance of the above theorem is just as $P_T$ was the stochastic counterpart of the randomly perturbed map $T_d$. $P_K \cdot P_T$ is the stochastic counterpart of closed-loop $T_d \circ (1 + K_d)$ with certain disturbance $d$. The analysis, including derivation of probability and entropy estimates, for finite Markov chains is particularly straightforward. In the limit of taking finer partitions (or letting $d \to 0$), one recovers the stabilization problem $T \circ (1 + K)$.

We close this section with an example.

**Example 4** Consider a linear expanding map:

$$z_{n+1} = a(z_n + u_n),$$

where $a = 2$ is the expansion rate. The perturbed closed-loop is given by

$$x_{n+1} = a_d(1 + k_d)(x_n),$$

where $a_d$ and $(1 + k_d)$ are perturbed versions of $a$ and possibly nonlinear control mapping $k$. We define this perturbed problem w.r.t a partition $\mathcal{X} = \{D_1, D_2, D_3\}$ with three cells $D_1 = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$, $D_2 = [-\frac{3\varepsilon}{2}, -\frac{\varepsilon}{2}]$, and $D_3 = [\frac{\varepsilon}{2}, \frac{3\varepsilon}{2}]$. On $\mathcal{X}$, the $3 \times 3$ sub-Markov matrix is

$$P_T = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (35)$$

With respect to the partition $\mathcal{X}$, the control objective is to “stabilize” the cell $D_1$ containing 0. On $\mathcal{X}$, denote by $P_K$ the Markov chain corresponding to control $(1 + k)$. The Markov chain for the closed-loop equation (34) is formally written as

$$\mu_{n+1} = \mu_n P_K \cdot P_T. \quad (36)$$

Note that Eq. (36) provides a *linear* description of Eq. (34) irrespective of whether $1 + k$ is linear, nonlinear, or even stochastic map. As an example, for

$$P_K = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ the closed-loop P-F is given by } P_K \cdot P_T = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}. \quad (37)$$

For the choice of $P_K$, the control may be a linear gain $k = -0.5$, a random gain where $k$ is a random variable chosen from set $[-0.5, -1]$ or a nonlinear quantization based feedback controller. Note that $a_d(1 + k_d) : X \to X$ (in Eq. (34)) for all these cases. The only invariant measure for $P_K \cdot P_T$ in this example is given by $[1/2, 1/4, 1/4]$. This implies that asymptotically a typical trajectory $\{x_n\}$ lies in the cell $D_1$ with probability 1/2, i.e.,

$$\text{Prob}(x_n \in D_1) \doteq \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_{D_1}(x_n) = \frac{1}{2}, \quad (38)$$

and the cell $D_1$ is not stable in the conventional sense.
IV. FUNDAMENTAL LIMITATIONS

In this paper, fundamental limitations are expressed within the stochastic framework, i.e., by replacing $T_d$ and a stabilizing $K_d$ by $P_T$ and $P_K$ respectively. It is assumed that the $T$ is an expanding dynamical system where the 0 equilibrium is unstable and furthermore all of the eigenvalues of its linearization are outside the unit circle. For such a problem, control-independent probability and conditional entropy estimates are obtained in the following two subsections respectively.

A. Probability bounds

In order to obtain probability bounds, it is useful to extend the notion of stability in terms of invariant measures of $P_K \cdot P_T$ as follows.

**Definition 5** Consider a closed-loop system $T_d \circ (1 + K_d) : X \rightarrow X$ together with its unique invariant measure $\mu$ with support $A \subset X$. A subset $S \subseteq A$ is $q$-stable if

$$\mu(S) = q. \tag{39}$$

A set $S$ is stable if it is $q$-stable with $q = 1$. Intuitively, $q$-stability provides for a weaker notion of stabilty whereby any “typical” trajectory of the dynamical system lies in set $S$ only a fraction (equal to $q$) of the time:

$$\text{Prob}(x_n \in S) \doteq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_S(x_n) = q, \tag{40}$$

For the stabilization problem with respect to a finite partition, suppose $\mathcal{X}_L = \{D_1,D_2,\ldots,D_L\}$ denotes a finite partition with a matrix $P_T$ whose first row denoted as

$$[P_T]_1 = [p_1, \ldots, p_L] \tag{41}$$

is assumed to be given. By virtue of the fact that $P_T$ is a Markov or a sub-Markov matrix, $p_i \geq 0$ and $\sum_i p_i \leq 1$. Additionally, assume that the first column denoted as

$$[P_T]^\top = [p_1, 0, \ldots]' \tag{42}$$

has atmost one non-zero entry $p_1$. The resulting Markov chain is drawn in Fig. 1. It is a model of a dynamical systems with an unstable equilibrium in cell $D_1$. For instance, the Markov chain for the example in Sec. III where $T(x) = 2x$ has as its first row $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and first column $[\frac{1}{2}, 0, 0]'$. The control objective is to design $P_K$ to $q$-stabilize $D_1$ with say maximal possible value of $q$. For this we have a following fundamental – control independent – result on the $q$-stabilization:
Theorem 6 Consider $P_T$ defined on a partition $\mathcal{X}_L$ with the structure in Fig. 1 and the first row and column given in Eqs. (41) and (42) respectively. Let $P_K$ denote a control Markov matrix on $\mathcal{X}_L$. If $\underline{\mu} = [\mu_1, \ldots]$ is an invariant measure for the closed-loop Markov matrix $P_K \cdot P_T$ then
\[ \mu_1 \leq p_1, \quad (43) \]
i.e., the maximum possible value for $q$-stabilization of $D_1$ is $q = p_1$. Suppose $D_1$ is made $p_1$-stable. Then
1) $\sum_{i=1}^{L} p_i = 1$,
2) For every $i$ with $p_i \neq 0$, the $i^{th}$ row of the control $[P_K]_i = [1,0,\ldots]$.
3) $\underline{\mu}$ with $\mu_j = p_j$ gives an invariant probability measure of the controlled markov chain $P_K \cdot P_T$.

Proof: Suppose $\underline{\mu} = \mu P_K \cdot P_T$. Due to the assumption on the first column of $P_T$ (see Eq. (42)),
\[ \mu_1 = <\underline{\mu}, [P_K]_1^T > p_1, \quad (44) \]
where $[P_K]_1$ denotes the first column of the Markov chain $P_K$ and $<,>$ denotes the standard inner product. If $\underline{\mu}$ is a probability measure then $\sum_i \mu_i = 1$ and
\[ \mu_1 \leq \sum_i \mu_i \cdot \max_i [P_K]_{1i} \cdot p_1 \leq p_1. \quad (45) \]
This gives the desired inequality (43). Now, suppose $P_K$ is such that $D_1$ is made $p_1$-stable. Using Eq. (44) together with the fact that $\sum_i \mu_i = 1$, we have
\[ \mu_1 = p_1 \text{ implies } [P_K]_{11} = 1 \text{ whenever } \mu_i \neq 0. \quad (46) \]
In particular, because $\mu_1 = p_1 > 0$, $[P_K]_{11} = 1$ and because $P_K$ is a Markov matrix, its first row $[P_K]_1 = [1,0,\ldots]$. This shows the part (2) for the particular case of $i = 1$. By matrix multiplication,
\[ [P_K \cdot P_T]_1 = [p_1,p_2,\ldots,p_L]. \quad (47) \]
As $P_K \cdot P_T$ is a Markov matrix, necessarily $\sum_i p_i = 1$. This shows the part (1). Since $\underline{\mu}$ is an invariant measure,
\[ [\mu_1, \mu_2, \ldots, \mu_L] \times \times \times = [\mu_1, \mu_2, \ldots, \mu_L], \quad (48) \]
which implies
\[ \mu_i = \mu_1 p_i + \ldots \geq p_1 p_i. \quad (49) \]
As a result, $\mu_i > 0$ for all $i$ such that $p_i > 0$. Using the condition in Eq. (46),
\[ [P_K]_i = [1,0,\ldots] \text{ whenever } p_i > 0. \quad (50) \]
This shows (2). Once again, by matrix multiplication
\[ [P_K \cdot P_T]_i = [p_1,p_2,\ldots,p_L] \text{ whenever } p_i > 0 \quad (51) \]
Finally, because of part (1) and Eq. (51), it is easy to verify that $\mu_j = p_j$ is an invariant probability measure for the Markov chain $P_K \cdot P_T$.

The theorem gives limitations on a) maximum achievable value of $q$ for $q$-stability of $D_1$, and b) the resulting invariant measure. Either of these are a function of only the properties of the open-loop Markov chain $P_T$. The key assumption needed for the conclusions is the structure of $P_T$ with respect to $D_1$ – as expressed by Eq. (42).
B. Entropy bounds

First consider a scalar linear dynamical system,

\[ z_{n+1} = b(z_n + u_n), \]

with the expansion rate \( a = |b| > 1 \). We will derive the limitations for a finite partition with cell \( D_1 = [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \) denoting the \( \epsilon \)-neighborhood of 0. For reasons that will become clear in the following, we take \( X = a(D_1) = [-a\frac{\epsilon}{2}, a\frac{\epsilon}{2}] \) to be the compact set. As before, \( \mathscr{R}_L = \{D_1,D_2,\ldots,D_L\} \) denotes a finite partition and \( \mathcal{D} = \{\partial(1),\ldots,\partial(L)\} \) the conditional dither. Associated with the discrete partition, the first row and column of the sub-Markov chain \( P_T \) (for the state evolution \( Tz = b(z) \)) are:

\[
[P_T]_1 = \begin{bmatrix}
\frac{1}{a}, & \frac{m(D_2)}{a\epsilon}, & \cdots, & \frac{m(D_L)}{a\epsilon} \\
0, & \cdots, & \cdots, & 0
\end{bmatrix},
\]

\[
[P_T]^1 = \begin{bmatrix}
\frac{1}{a}, & 0, & \cdots, & 0
\end{bmatrix},
\]

respectively. With respect to the partition, we consider the perturbed closed-loop equation

\[ x_{n+1} = b_d \circ (1 + k_d)x_n, \]

where 1) \( k(\cdot) \) is any stabilizing control, and 2) the disturbance \( \{d_n\} \) arises due to the conditional dither \( \mathcal{D} \). The stochastic analogue of Eq. (54) is the now familiar

\[ \mu_{n+1} = \mu_n P_K P_T, \]

where \( P_K \) is the Markov chain constructed from the control \( k \). With discrete Markov chains, the stabilization problem was posed as the \( q \)-stabilization of cell \( D_1 \). The following lemma provides the relationship between this and the original problem.

**Lemma 7** Suppose \( u_n = k(x_n) \) be any linear stabilizing control of Eq. (52) and \( \mathscr{R}_L \) be a finite partition with conditional dither \( \mathcal{D} \). Then for the closed-loop Eq. (54), the cell \( D_1 \) is \( q \)-stable with maximal value of \( q = \frac{1}{a} \).

**Proof:** The condition for closed-loop stability is

\[ |a(1+k)| < 1 \]

which necessarily implies that \( (1+k):a(D_i) \rightarrow D_1 \) for \( i = 1,\ldots,L \). The Markov chain for the control and the closed-loop are then given by

\[ P_K = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}, \quad P_K \cdot P_T = \begin{bmatrix}
\frac{1}{a}, & \frac{m(D_2)}{a\epsilon}, & \cdots, & \frac{m(D_L)}{a\epsilon} \\
0, & \cdots, & \cdots, & 0
\end{bmatrix}
\]

respectively and its closed-loop invariant measure is

\[ \mu = \begin{bmatrix}
\frac{1}{a}, & \frac{m(D_2)}{a\epsilon}, & \cdots, & \frac{m(D_L)}{a\epsilon}
\end{bmatrix}, \]

i.e., the cell \( D_1 \) is \( \frac{1}{a} \)-stable. Using Theorem 6, this is also the maximum value possible. Using Theorem 3, the random sequence \( \{x_n\} \) for the perturbed closed-loop (54) lies in the cell \( D_1 \) with probability \( \frac{1}{a} \). The inability to \( q \)-stabilize the cell \( D_1 \) for arbitrary value of \( q \) (an in particular, for \( q = 1 \)) constitutes a fundamental limitation in stabilization. This depends only upon the open-loop dynamics, expansion rate \( a \) here,
and is independent of the choice of the feedback control gain $k$. Conversely, as the lemma shows, any stabilizing control achieves the upper bound $\frac{1}{a}$. Next, larger the value of $a$, the larger the uncertainty in the state $\{x_n\}$ of closed-loop system – it could be anywhere in cell $a(D_1)$ of length $a \cdot \varepsilon$. This uncertainty is best expressed in terms of the entropy metric and relates to the Bode formula. In order to make this correspondence precise, we need to define entropy rate for the disturbance. One possibility is to take it to be the entropy rate for the conditional dither acting on the control map $(1 + k_d)$:

$$H(d) = H(\partial(1)) = \ln(\varepsilon).$$  

This choice is dictated by our choice of cell $D_1$ (with volume $\varepsilon$) and the fact that partition of $a(D_1) \setminus D_1$ is taken to be arbitrary. In effect, the conditional dither in cell $D_1$ determines the uncertainty of the output $x$ and one obtains a version of Bode formula:

**Theorem 8** Consider the perturbed closed-loop Eq. (54) with the expansion rate $a > 1$. For any stabilizing control gain $k$, the random output sequence $\{x_n\}$ has entropy given by

$$H_c(x_n) = \ln(a) + H(d).$$  

*Proof:* Using Theorem 6 and the expression for the invariant measure $\mu$ in Eq. (58),

$$\text{Prob}(x_n \in D_1) = \frac{1}{a}$$

$$\text{Prob}(x_n \in D_i) = \frac{m(D_i)}{a \varepsilon} \quad \text{for} \quad i = 2,\ldots,L.$$  

$x_n$ is uniformly distributed within each cell and it follows that the stationary pdf of $x_n$ is

$$f(x) = \begin{cases} \frac{1}{a \varepsilon} & \text{for} \quad x \in D_1, \\ \frac{m(D_i)}{a \varepsilon m(D_i)} = \frac{1}{a \varepsilon} & \text{for} \quad x \in X - D_1, \end{cases}$$  

i.e., $f(x) = \frac{1}{a \varepsilon}$ is the uniform distribution for $x \in a(D_1)$. Next, using Eq. (57) and a similar calculation as above, the conditional pdf for the stationary case is

$$f(x_2|x_1) = \begin{cases} \frac{1}{a \varepsilon} & \text{for} \quad x_2 \in D_1, \\ \frac{m(D_j)}{a \varepsilon m(D_j)} = \frac{1}{a \varepsilon} & \text{for} \quad x_2 \in X - D_1 \end{cases}$$  

i.e., $f(x_2|x_1) = \frac{1}{a \varepsilon}$ for $x_2,x_1 \in X$. Now, applying the formula for relative entropy, we have

$$H(x_2|x_1) = \int_X f(x_1) \int_X f(x_2|x_1) \ln(f(x_2|x_1)) dx_2 dx_1$$

$$= \ln(a) + \ln(\varepsilon),$$  

where $\ln(\varepsilon) = H(d)$ is the entropy of the conditional dither in cell $D_1$. The proof is completed by noting that $x_n$ depends only upon $x_{n-1}$ and not its entire history, i.e., $\{x_n\}$ is a Markov process. For a stationary Markov process, the conditional entropy

$$H(x_n|x_{n-1},\ldots,x_{n-m}) = H(x_n|x_{n-1}) = H(x_2|x_1),$$  

(65)
where the first equality is due to the Markov assumption and the second equality is due to the stationarity [2].

**Remark 9** We assumed a finite partition for \( X = a(D_1) \) because it represents the support of the invariant measure with conditional dither in cell \( D_1 \) and a stabilizing controller. Additional cells taken from a partition over any larger (than \( a(D_1) \)) set will not change the results as the relative entropy is calculated with respect to the invariant measure. Next, the estimate holds for an arbitrary partition of \( a(D_1) \setminus D_1 \). Thus, one can express the result with respect to an partition \( \{D_1, D_2, \ldots \} \) where \( \{D_2, \ldots \} \) is only assumed to be sufficiently fine.

With conditional dither, it is more convenient to express a formula for the relative entropy directly in terms of entries of the Markov matrix as summarized in the following lemma.

**Lemma 10** Consider a closed-loop Eq. (54) together with its associated Markov chain \( P = P_K \cdot P_T \) on \( \mathcal{X}_L = \{D_1, \ldots, D_L\} \). Let \( \{\mu_i\} \) be the invariant measure of \( P \). Then the sequence \( \{x_n\} \) for the closed-loop Eq. (54) has entropy

\[
H_c(x_n) = -\sum_{i,j=1}^L \mu_i P_{ij} \ln(P_{ij}) + \sum_{i,j=1}^L \mu_i P_{ij} \ln(m(D_j)),
\]

where \( m(D_i) \) is the volume of cell \( D_i \). If \( P \) comprises of repeated rows \( [P]_j = [p_1, \ldots, p_L] \) for \( j = 1, \ldots, L \) then

\[
H_c(x_n) = -\sum_{j=1}^L p_j \ln\left( \frac{p_j}{m(D_j)} \right).
\]

**Proof:** Eq. (66) follows from using the formula for the relative entropy and the relationships:

\[
f(x_1) = \frac{\mu_i}{m(D_i)} \quad \text{for} \quad x_1 \in D_i,
\]

\[
f(x_2|x_1) = \frac{P_{ij}}{m(D_j)} \quad \text{for} \quad x_1 \in D_i, \ x_2 \in D_j.
\]

Eq. (67) then follows as a special case.

Denoting \( s_1 \equiv (1 + k_d)x_1, \ y \) to be its invariant measure, \( P = P_T \), and assuming stationarity, we also note that

\[
H(x_2|x_1) = -\sum_{i,j} v_{ij} P_{ij} \ln\left( \frac{P_{ij}}{m(D_j)} \right).
\]

This formula will be useful latter.

We note that the proof of Theorem 8 did not use linearity of maps \( b \) and \( (1 + k) \). In fact, we did not even use these mappings, rather only the Markov chains \( P_K \) and \( P_T \). Below, we use this framework for the general multi-state expanding nonlinear dynamical system:

\[
\text{state : } \ z_{n+1} = T(z_n + u_n),
\]

where the Jacobian \( DT(0) \) has only unstable eigenvalues \( |\lambda_i| > 1 \) for \( i = 1, \ldots, m \), and expansion rate \( a = |DT(0)| > 1 \). In order to apply the framework, let \( K \) be a stabilizing control and \( X \) be \( O(\varepsilon) \)-neighborhood of 0 with the property that

\[
T \circ (1 + K) : X \to X.
\]

Now, for the multi-state case, it is not necessary that a stabilizing control \( (1 + K) : X \to X \), a condition that has been assumed thusfar. Denote \( Y \equiv (1 + K)(X) \) so

\[
(1 + K) : X \to Y.
\]
Note that $0 \in Y$ and $m(Y) = O(\varepsilon)$. Eqs. (71)-(72) imply that $T : Y \to X$. Consider next a finite partition of $Y$ and $X$ denoted as $\mathcal{Y}_M = \{E_1, \ldots, E_M\}$ and $\mathcal{Y}_L = \{D_1, \ldots, D_L\}$ to write the perturbed equations:

$$(1 + K_d) : X \to Y, \quad T_d : Y \to X, \quad x_{n+1} = T_d \circ (1 + K_d)(x_n).$$

(73)

We assume $D_1$ to be the cell containing $0$. Next, express

$$x_{n+1} = T_d(s_n), \quad \text{where} \quad s_n = (1 + K_d)x_n$$

(74)

represents a stabilizing control input. Since $x_1 \to s_1 \to x_2$ form a Markov chain,

$$H(x_2|x_1) \geq H(x_2|s_1)$$

(75)

because of the data processing inequality; see section 2.8 of [2]. The right-hand side of Eq. (75) is estimated using the Markov chain formalism. In particular, let $\mu$ be the invariant measure of $\{x_n\}$, and $\nu$ be the invariant measure of control input $\{s_n\}$. In the following $P$ indicates the Markov chain $P_T$ for the dynamical system $T$. By Lemma 2 and Eq. (74), these measures are related as

$$\mu_j = \sum_{i=1}^{M} v_i P_{ij} \quad \text{for} \quad j = 1, \ldots, L.$$ 

(76)

By taking a summation $\sum_{j=1}^{L}$ for Eq. (76), $\sum_{j=1}^{L} P_{ij} = 1$ for $i = 1, \ldots, M$. With this notation, Eq. (69) gives the formula for

$$H(x_2|s_1) = - \sum_{i=1}^{M} \sum_{j=1}^{L} v_i P_{ij} \ln\left(\frac{P_{ij}}{m(D_j)}\right)$$

(77)

In the limit that $\varepsilon \to 0$, the individual entries $P_{ij}$ are related to the local expansion rate as follows:

$$\lim_{\varepsilon \to 0} P_{ij} = \lim_{\varepsilon \to 0} \frac{1}{m(E_i)} \int_{E_i} \chi_{T^n}(D_j) dm(x)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{m(E_i)} \int_{T^n(E_i)} \chi_j(D_j)|DT^{-1}(y)| dm(y) = \frac{1}{a} \lim_{a \to 0} \frac{m(T(E_i) \cap D_j)}{m(E_i)} \ln\left(\frac{P_{ij}}{m(D_j)}\right),$$

(78)

because $\lim_{\varepsilon \to 0} |DT^{-1}(\cdot)| = \frac{1}{a}$. Since, entropy rates are computed with respect to a (given partition) of $X$, we take $M = 1$ and $\mathcal{Y}_i = \{Y\}$ to be a singleton and re-scale volumes so $m(Y) = \varepsilon$; an estimate with a general partition $\mathcal{Y}$ is also provided. The entropy of the disturbance is taken to be for the dither defined w.r.t $\mathcal{Y}$; for $\mathcal{Y}_i$, $H(d) = \ln(\varepsilon)$.

**Theorem 11** Consider an expanding dynamical system given by Eq. (70) with $a = |DT(0)|$. Let $K$ be a any stabilizing control such that $DT(0) : (1 + DK)(0)$ has eigenvalues inside the unit circle. Then in the limit of vanishing conditional dither ($\varepsilon \to 0$), the output sequence $\{x_n\}$ of the closed loop dynamical system $T_d \circ (1 + K_d)$ has entropy

$$H_e(x_n) \geq \ln(a) + H(d).$$

(79)

Next, assume the control $K$ to be **strongly** stabilizing the cell $D_1$, i.e., $(1 + K) : T(D_1) \to D_1$. Then as $\varepsilon \to 0$,

$$H_e(x_n) = \ln(a) + H(d).$$

(80)

**Proof:** Using the estimate in Eq. (78),

$$\frac{P_{ij}}{m(D_j)} = \frac{1}{a \cdot m(E_i)} \frac{m(T(E_i) \cap D_j)}{m(D_j)} = \frac{1}{a \cdot m(E_i)} Q_{ij},$$

(81)
where $Q_{ij} \leq 1$. Substituting this in Eq. (77),

$$H(x_2|s_1) = -\sum_{i,j=1}^{M} v_i P_j \ln(Q_{ij}) + \sum_{i=1}^{M} \sum_{j=1}^{L} v_i P_j \ln(a \cdot m(E_i))$$

$$\geq \ln(a) + \sum_{i,j} v_i P_j \ln(m(E_i)).$$

Finally using the fact that $\sum_j P_{ij} = 1$ and

$$H(x_2|s_1) \geq \ln(a) + \sum_{i=1}^{M} v_i \ln(m(E_i)).$$

The term $\sum_i v_i \ln(m(E_i))$ corresponds to the entropy of the conditional dither defined with respect to the partition $\{E_1, \ldots, E_M\}$. Specifically with $M = 1$,

$$H(x_2|s_1) \geq \ln(a) + \ln(\epsilon) = \ln(a) + H(d).$$

With strong stabilization, $Y = D_1$, $X = T(D_1)$, and $P_K$ is given by Eq. (57). We derive the entropy estimate with the simplest two cell partition $\mathcal{A}_2 = \{D_1, D_2\}$ where $D_2 = T(D_1) \setminus D_1$. For such a partition, the closed-loop Markov matrix

$$P_K \cdot P_T = \begin{bmatrix} q^\epsilon & 1 - q^\epsilon \\ q^\epsilon & 1 - q^\epsilon \end{bmatrix},$$

where $q^\epsilon = [P_T]_{11}$; the superscript makes the dependence on $\epsilon$ explicit. Using the formula (67), one obtains an entropy estimate as

$$H_c(x_n) = -q^\epsilon \ln(q^\epsilon) - (1 - q^\epsilon) \ln \left( \frac{1 - q^\epsilon}{m(D_2^c)} \right).$$

To prove the result, we show that

$$\lim_{\epsilon \to 0} q^\epsilon = \frac{1}{a}, \quad \lim_{\epsilon \to 0} \frac{m(D_2^c)}{\epsilon} = (a - 1).$$

Indeed, the latter equation is clear by construction and in the $\epsilon = 0$ limit,

$$q^0 = \lim_{\epsilon \to 0} \frac{m(T^{-1}D_1 \setminus D_1)}{m(D_1)} = \lim_{\epsilon \to 0} \frac{m(T^{-1}D_1)}{m(D_1)} = |DT^{-1}(0)| = \frac{1}{a},$$

where $a = |DT(0)|$ is the expansion rate. Since the argument relies on a type of linearization, the estimate is insensitive to the choice of partition in the limiting case. In particular, a finer partition of $T(D_1) \setminus D_1$ will also yield the same conclusion.(see also Theorem 8 and Remark 9).

As a result of this theorem, the stabilization in the general multi-state case only leads to an inequality for the entropy estimate. With strong stabilization, one obtains an equality. Note that strong stabilization is equivalent to stabilization for the scalar case. The general multi-state result was first shown with the aid of TFE in Nair et. al. [7] who also showed the bound to be tight with strong stabilization. Our result is motivated by [7] and is analogous with one important difference: using TFE, the uncertainty arises due to the initial condition while using conditional dither, the uncertainty arises due to disturbance. After taking appropriate limits in each case (finer partitions or vanishing conditional dither), one obtains a bound in terms of expansion rate $a$ due to unstable dynamics. We further remark that the notion of stabilization needed to obtain these bounds is weaker than Lyapunov stability. The control $K$ need be stabilizing only in the sense that Eqs. (71)-(72) are valid, i.e., the closed-loop map is positively invariant for certain small neighborhood of 0. Finally, we stress the important
role of disturbance. At each point, one is indeed solving a perturbed problem where \( \ln(\varepsilon) \) denotes the entropy rate due to conditional dither. In order to interpret the \( \varepsilon \to 0 \) limit, we note that the output \( x_n \) is a continuous random variable. Thus, it is meaningful to talk only about change in uncertainty and this change is reflected here as difference between uncertainty rates of \( x_n \) and \( d_n \) given by the Bode formula (79) for the general case and (80) for the scalar or the strong stabilization case.

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