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Computation of Lyapunov measure for almost everywhere stability

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Abstract— **In our recent paper [1],** *Lyapunov measure* **is introduced as a new tool for verifying almost everywhere stability of an invariant set in a nonlinear dynamical system or continuous mapping. It is shown that for almost everywhere stable system explicit formula for the Lyapunov measure can be obtained as a infinite series or as a resolvent of stochastic linear operator. This paper focus on the computation aspects of the Lyapunov measure. Methods for computing these Lyapunov measures are presented based upon set-oriented numerical approaches, which are used for the finite dimensional approximation of the linear operator. Stability results for the finite dimensional approximation of the linear operator are presented. The stability in finite dimensional space results in further weaker notion of stability which in this paper is referred to as coarse stability.**

I. INTRODUCTION

In nonlinear dynamical systems Lyapunov function based methods play a central role in carrying out stability analysis and control synthesis. However as opposed to linear systems there are very few computational algorithm [2] to construct these functions in general nonlinear systems. This forms an important barrier for the wide spread use of Lyapunov based methods in nonlinear system theory. This paper is an effort towards overcoming this barrier.

In [3], Rantzer's introduced a weaker notion of almost everywhere stability. Density function is proposed for the verification of almost everywhere stability. Existence of density function guarantee global attractive property of the equilibrium solution from almost every with respect to Lebesgue measure initial condition in the phase space. This density function is shown to be dual to the Lyapunov function. In our paper [1], Lyapunov measure is proposed for verifying almost everywhere stability of an invariant set for dynamical systems or continuous mapping. Lyapunov measure is also shown to be dual to Lyapunov function. This duality between Lyapunov function and Lyapunov measure is shown to be connected to the dual nature of two stochastic linear operators called as Koopman and Frobenius-Perron (P-F) operators [4]. For stable and almost everywhere stable system, explicit formulas for the Lyapunov function and Lyapunov measure were obtained in terms of resolvent of Koopman and Frobenius-Perron operator respectively.

Koopman and Frobenius-Perron operators are used in the study of transport properties of ODE, dynamical systems or continuous mapping. While dynamical system describe the evolution of single trajectory, Koopman and Frobenius-Perron operators are used to study the evolution of ensembles of trajectories. Most recently there has been a significant interest in the applied dynamical system community to develop a computational tool for the study of global dynamics using these operators [5]. In particular set-oriented numerical methods have recently been used to approximate the infinite-dimensional stochastic linear operator using their finite-dimensional counterpart [6]. These tools have been used in the study of approximation and visualization of complex behavior and invariant sets [5], [7], for transport problem in celestial mechanics and chemistry [8], [9] and for the comparison of complex behavior [10], [11].

Almost everywhere stability results for the infinite dimensional Frobenius Perron operator in terms of existence of Lyapunov measure are proved in [1]. In this paper we present the stability result for the finite dimensional approximation of the infinite dimensional Frobenius Perron operator. The result are motivated from the computation view in mind. Interestingly the stability in finite dimensional case result in further weaker notion of almost everywhere stability which in this paper is referred to as coarse stability. The explicit formula for the Lyapunov measure in the infinite dimensional case provides us with various methods for the computation of Lyapunov measure in finite dimensional space. For the computation of the finite dimensional approximation of the Lyapunov measure we used set oriented numerical approaches developed in ([5], [6])

The outline of this paper is as follows. In Section II, we summarize some of the key stability result from [1] for the infinite dimensional P-F operator. In Sections III we present set-oriented numerical methods for the finite dimensional approximation of the infinite dimensional P-F operator. In section IV, we prove result on the coarse stability in finite dimensional case and give different formulas for the computation of Lyapunov measure in finite dimensional case. Examples are presented in section V and conclusion and discussion follows in section VI

II. STABILITY IN INFINITE DIMENSION

In this paper, *discrete dynamical systems* or mappings of the form

$$x_{n+1} = T(x_n) \quad (1)$$

are considered. $T : X \to X$, where $X \subset \mathbb{R}^n$ is a compact set. $\mathscr{B}(X)$ denotes the Borel σ-algebra on X and $\mathscr{M}(X)$ the vector space of bounded real valued measures on $\mathscr{B}(X)$. The mapping T is assumed to be continuous and non-singular. The mapping T is said to be non-singular with respect to measure $\mu \in \mathscr{M}(X)$ if $\mu(T^{-1}B) = 0$ for all $B \in \mathscr{B}(X)$ such that $\mu(B) = 0$. The stochastic P-F operator for a mapping

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$T : X \to X$ is given by

$$\mathbb{P}[\mu](A) = \mu(T^{-1}(A)), \tag{2}$$

where $\mu \in \mathscr{M}(X)$ and $A \in \mathscr{B}(X)$. The invariant measure are the fixed points of the P-F operator $\mathbb{P}$ that are additionally probability measures. A set $A \subset X$ is said to be invariant for the dynamical system T if $T(A) = A$. We use following two definitions for the stability of the invariant set A

Definition 1 (Almost everywhere (a.e.) stable): An invariant set A for the dynamical system $T : X \to X$ is said to be stable almost everywhere (a.e.) with respect to finite measure $m \in \mathscr{M}(A^c)$ if

$$m\{x \in A^c : \omega(x) \notin A\} = 0 \tag{3}$$

where $\omega(x)$ is the $\omega-$ limit set of A [12].

Definition 2 (a.e. stable with geometric decay): The invariant set $A \subset X$ for the dynamical system $T : X \to X$ is said to be stable almost everywhere with geometric decay w.r.t. to a finite measure $m \in \mathscr{M}(A^c)$ if given $\varepsilon > 0$, there exists $K(\varepsilon) < \infty$ and $\beta < 1$ such that

$$m\{x \in A^c : T^n(x) \in B\} < K\beta^n \quad \forall\, n \geq 0 \tag{4}$$

for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$, where $U(\varepsilon)$ is the ε neighborhood of the invariant set A for $\varepsilon > 0$.

It can be shown that if A is an invariant set for the dynamical system T, then the P-F operator associated with T admits following lower triangular decomposition [1].

$$\mathbb{P} = \begin{pmatrix} \mathbb{P}_0 & 0 \\ \times & \mathbb{P}_1 \end{pmatrix} \tag{5}$$

where $\mathbb{P}_0 : \mathscr{M}(A) \to \mathscr{M}(A)$ is the Markov operator and $\mathbb{P}_1 : \mathscr{M}(A^c) \to \mathscr{M}(A^c)$ is the sub-Markov operator. Using this decomposition, we give following definition of Lyapunov measure in terms of sub-Markov operator $\mathbb{P}_1$.

Definition 3 (Lyapunov measure): is any non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$, which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$ and satisfies

$$\mathbb{P}_1\bar{\mu}(B) < \alpha\bar{\mu}(B), \tag{6}$$

for every set $B \subset \mathscr{B}(A^c)$ with $\bar{\mu}(B) > 0$ and $\alpha \leq 1$ is some positive constant.

Now we state the main result on almost everywhere stability of the invariant set A in terms of existence of Lyapunov measure.

Theorem 4: Consider $T : X \to X$ in Eq. (1) with an invariant set $A \subset X$ and $A^c = X \setminus A$. Suppose there exists a Lyapunov measure $\bar{\mu}$ with $\alpha = 1(\alpha < 1)$, then the invariant set A is almost everywhere stable (with geometric decay) w.r.t. to any absolutely continuous measure $m \prec \bar{\mu}$.

For proof refer [1]. This in brief summarize the stability result for the infinite dimensional P-F operator as appear in [1]. In the remaining sections, we study how these result can be implemented in the finite dimensional approximation of the P-F operator.

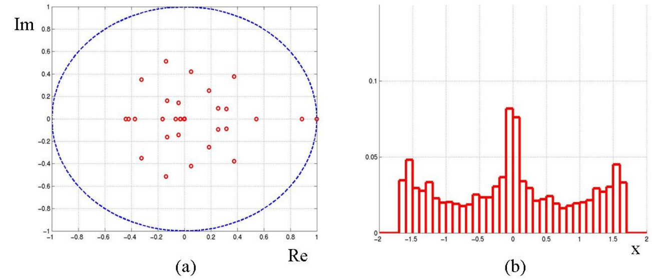

Fig. 1. (a) Eigenvalues and (b) the invariant measure of the discretized P-F matrix for the logistic map

III. DISCRETIZATION OF THE P-F OPERATOR

In order to obtain a finite-dimensional (discrete) approximation of the continuous P-F operator, one considers a finite partition of the phase space X, denoted as

$$X_L \doteq \{D_1, \cdots, D_L\}, \tag{7}$$

where $\cup_j D_j = X$. These partitions can be constructed by taking quantization for states in X. Instead of a Borel σ-algebra, consider now a σ-algebra of the all possible subsets of X_L. A real-valued measure μ_j is defined by ascribing to each element D_j a real number. Thus, one identifies the associated measure space with a finite-dimensional real vector space $\mathbb{R}^L$. In particular for $\mu = (\mu_1, ..., \mu_L) \in \mathbb{R}^L$ define a measure on X as

$$d\mu(x) = \sum_{i=1}^{L} \mu_i \kappa_i(x) \frac{dm(x)}{m(D_i)} \tag{8}$$

where m is the Lebesgue measure and κ_i denotes the indicator function with support on set D_i. The discrete P-F approximation arises as a matrix on this "measure space" $\mathbb{R}^L$ and is given by

$$P_{ij} = \frac{m(T^{-1}(D_j) \cap D_i)}{m(D_i)}, \tag{9}$$

m being the Lebesgue measure. The resulting matrix is non-negative and because $T : D_i \to X$, $\sum_{j=1}^{L} P_{ij} = 1$ i.e., P is a Markov or a row-stochastic matrix.

The finite-dimensional Markov matrix P is used to numerically study the approximate asymptotic dynamics of the Dynamical system T; cf., [13], [5]. In particular, suppose $\mu \geq 0$ is an invariant probability measure (vector), i.e.,

$$P\mu = 1 \cdot \mu, \tag{10}$$

such that $\sum \mu_i = 1$ then the support of μ gives the approximation of the attractor and $\mu_i = \mu(D_i)$ gives the "weight" of the component D_i in attractor A [14].

As an example, Figure 1 depicts the spectrum of the P-F operator and the invariant measure for the logistic map $T(x;\lambda) = \lambda x - x^3$, where $\lambda = \frac{3}{2}\sqrt{3} + 10^{-2}$. The support of the invariant measure captures the the chaotic attractor of the logistic map.

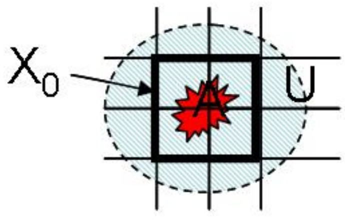

Fig. 2. A schematic of the three sets $A \subset X_0 \subset U$: A denotes the attractor set, X_0 is the support of its invariant measure approximation, and U is some neighborhood. The finite partition is shown as the rectangular grid in the background.

IV. STABILITY IN FINITE-DIMENSION

In this section, discretization methods are used to approximate the Lyapunov measure. The existence of an approximation is related to yet weaker notions of stability, termed as coarse stability.

A. Matrix decomposition

We begin by presenting a decomposition result for the approximation P corresponding to a finite partition. It is assumed that an approximation μ_0, to the physical measure μ of an attractor set $A \subset X$, has been computed by evaluating a fixed-point the matrix P. An indexing is chosen such that the two non-empty complementary partitions

$$\mathscr{X}_0 = \{D_1, ..., D_K\}, \quad \mathscr{X}_1 = \{D_{K+1}, ..., D_L\} \tag{11}$$

with domains $X_0 = \cup_{j=1}^{K} D_j$ and $X_1 = \cup_{j=K+1}^{L} D_j$ distinguish the approximation of the attractor set from its complement set respectively. In particular, $A \subset X_0$, μ_0 is supported and non-zero on $\mathscr{X}_0$, and one is interested in stability w.r.t the initial conditions in the complement X_1. For an attractor A with a physical measure defined w.r.t a neighborhood $U \supset A$, such sets exist for a sufficiently fine partition such that $A \subset X_0 \subset U$; cf., Figure 2. The following Lemma summarizes the matrix decomposition result.

Lemma 5: Let P denote the Markov matrix for the mapping T in Eq. (1) defined w.r.t the finite partition $\mathscr{X}$ in Eq. (7). Let $M \cong \mathbb{R}^L$ denote the associated measure space and μ denote a given invariant vector of P. Suppose $\mathscr{X}_0$ and $\mathscr{X}_1$ are the two *non-empty* components as in Eq. (11) defined w.r.t μ such that $\mu > 0$ on $\mathscr{X}_0$; $\mu_i > 0$ iff $D_i \in \mathscr{X}_0$. Let $M_0 \cong \mathbb{R}^K$ and $M_1 \cong \mathbb{R}^{L-K}$ be the measure spaces associated with $\mathscr{X}_0$ and $\mathscr{X}_1$ respectively. Then for the splitting $M = M_0 \oplus M_1$, the P matrix has a lower triangular representation

$$P = \begin{bmatrix} P_0 & 0 \\ \times & P_1 \end{bmatrix} \tag{12}$$

where $P_0 : M_0 \to M_0$ is the Markov matrix with row sum equal to one and $P_1 : M_1 \to M_1$ is the sub-Markov matrix with row sum less than or equal to one.

Proof: Refer [15] for the proof ■

Our strategy is to study the stability in terms of properties of the matrix P_1 and define coarser (weaker) notions of stability with respect to initial conditions corresponding to this.

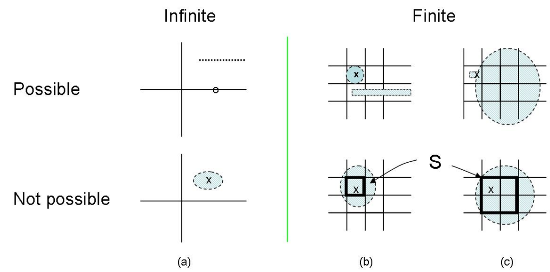

Fig. 3. A schematic comparing a.e. stability in infinite-dimensional setting (part (a)) to the coarse stability with finite partitions (part (b) and (c)). In either case, appropriate notion of stochastic stability is assumed ($\mathbb{P}_1$ and P_1 transient).

B. Coarse stability

In this section ,stability of the finite dimensional approximation of P-F operator is expressed in terms of the transient properties of the stochastic matrix P_1.

Definition 6 (Transient states): A sub-Markov matrix P_1 has only **transient states** if $P_1^n \to 0$, element-wise, as $n \to \infty$. Intuitively, it makes sense that transience be necessary given stability or a.e. stability w.r.t initial conditions. Conversely, transience is shown to imply yet weaker forms of stability referred to as **coarse stability** in this paper.

Definition 7 (Coarse Stability): Consider an attractor $A \subset X_0$ together with a finite partition $\mathscr{X}_1$ of the complement set $X_1 = X - X_0$. A is said to be **coarse stable** w.r.t the initial conditions in X_1 if for an attractor set $B \subset U \subset X_1$, there exists no sub-partition $\mathscr{S} = \{D_{s_1}, D_{s_2}, \dots, D_{s_l}\}$ in $\mathscr{X}_1$ with domain $S = \cup_{k=1}^{l} D_{s_k}$ such that $B \subset S \subset U$ and $T(S) \subseteq S$.
For *typical* partitions, coarse stability means stability modulo attractor sets B with domain of attraction U smaller than the size of cells within the partition. In the infinite-dimensional limit, where the cell size (measure) goes to zero, one obtains stability modulo attractor sets with measure 0 domain of attraction, i.e., a.e. stability. Figure 3 compares some of the possibilities with a.e. stability in infinite-dimensional settings and coarse stability using finite partitions. The part (a) shows that measure 0 invariant sets such as unstable equilibrium (denoted by o) or a (dashed) line in the plane may arise in the complement X_1 even with a.e. stability. However, stable equilibrium with a domain of attraction of positive measure is ruled out. The parts (b) and (c) consider coarse stability in discrete settings with a rectangular partition in the background. The part (b) shows that a stable equilibrium (denoted by x) or an elongated attractor set with a smaller, than cell size, domain of attraction is possible with coarse stability. However, an attractor whose domain of attraction contains a sub-partition S (marked with bold lines in the Fig. 3) in the complement set is not possible. In particular, coarse stability rules out the case where the cell containing a stable equilibrium itself lies in its domain of attraction. The part (c) shows that it is possible to construct a partition where coarse stability holds, yet the domain of attraction is very large w.r.t the partition. This is because the cell containing the stable equilibrium is not itself contained in the domain

of its attraction. We believe this to be atypical for reasonable choices of *fine enough* finite partition with the lower figure in part (c) being a better representative. Nevertheless, the scale of partition is important in deducing stability. The theorem below formally links the transience of matrix P_1 to various notions of stability considered in this paper. Before stating the theorem and its proof, we present a simple Lemma that is needed in the proof.

Lemma 8: Consider two equivalent measures $\mu \approx m$, and two sets S_1, S with $S_1 \subset S$. Then

$$\mu(S_1) = \mu(S) \quad \Leftrightarrow \quad m(S_1) = m(S). \tag{13}$$

Proof: Denote $S_1^c \doteq S \setminus S_1$ to be the complement set. We have, $\mu(S_1) = \mu(S) \Leftrightarrow \mu(S_1^c) = 0 \Leftrightarrow m(S_1^c) = 0 \Leftrightarrow m(S_1) = m(S)$. ∎

Theorem 9: Assume the notation of the Lemma 5. In particular, A is an attractor set in $X_0 \subset X$ with approximate invariant measure supported on the finite partition $\mathscr{X}_0$ of X_0. $\mathbb{P}_1$ is the sub-Markov operator on $\mathscr{M}(A^c)$. P_1 is its finite-dimensional sub-Markov matrix approximation obtained with respect to the partition $\mathscr{X}_1$ of the complement set $X_1 = X \setminus X_0$. For this

1) Suppose a Lyapunov measure $\bar{\mu}$ exists such that $\mathbb{P}_1\bar{\mu}(B) < \bar{\mu}(B)$ for all $B \subset \mathscr{B}(X_1)$, and additionally $\bar{\mu} \approx m$, the Lebesgue measure. Then the finite-dimensional approximation P_1 is transient.
2) Suppose P_1 is transient then A is coarse stable w.r.t the initial conditions in X_1.

Proof: 1. We first present a proof for the simplest case where the partition $\mathscr{X}_1$ consists of precisely one cell, i.e., $\mathscr{X}_1 = \{D_L\}$. In this case, $P_1 \in [0,1]$ is a scalar given by

$$P_1 = \frac{m(T^{-1}(D_L) \cap D_L)}{m(D_L)}, \tag{14}$$

where m is the Lebesgue measure. We need to show that $P_1 < 1$. Denote,

$$S = \{D_L\}, \quad S_1 = \{x \in D_L :\ T(x) \in D_L\}. \tag{15}$$

Clearly, $S_1 \subset S$ and existence of Lyapunov measure $\bar{\mu}$ satisfying Eq. (1) implies that

$$\bar{\mu}(S_1) = \mathbb{P}_1\bar{\mu}(S) < \bar{\mu}(S). \tag{16}$$

Using Lemma 8, $m(S_1) \neq m(S)$ and since $S_1 \subset S$, we have $m(S_1) < m(S)$. Using Eqs. (14) and (15), this implies $P_1 < 1$, i.e., P_1 is transient. We prove the result for the general case, where $\mathscr{X}_1$ is a finite partition, by contradiction. Suppose P_1 is not transient. Then using either the following Theorem 10, or a general result from the theory of finite Markov chains [16], [17], there exists atleast one non-negative invariant probability vector ν such that

$$\nu \cdot P_1 = \nu. \tag{17}$$

$$\text{Let} \quad S = \{x \in D_i :\ \nu_i > 0\}, \quad S_1 = \{x \in S : T(x) \in S\}. \tag{18}$$

$$\text{Claim :} \quad m(S_1) = m(S). \tag{19}$$

We first assume the claim to be true and show the desired contradiction. Clearly, $S_1 \subset S$ and if the claim were true, Lemma 8 shows that

$$\bar{\mu}(S_1) = \bar{\mu}(S). \tag{20}$$

Next, because $S \subset X_1$,

$$\mathbb{P}_1\bar{\mu}(S) = \bar{\mu}(T^{-1}(S) \cap X_1) \geq \bar{\mu}(T^{-1}(S) \cap S). \tag{21}$$

and this together with Eq. (20) gives $\mathbb{P}_1\bar{\mu}(S) \geq \bar{\mu}(S)$ for a set S with positive Lebesgue measure. This contradicts Eq. (1) and proves the theorem.

It remains to show the claim. Let $\{i_k\}_{k=1}^{l}$ be the indices with $\nu_{i_k} > 0$. Eq. (17) gives

$$\sum_{k=1}^{l} \nu_{i_k} [P_1]_{i_k j_m} = \nu_{j_m} \ \text{ for } \ m = 1, \ldots, l. \tag{22}$$

Taking a summation $\sum_{m=1}^{l}$ on either side gives

$$\sum_{k=1}^{l} \nu_{i_k} \sum_{m=1}^{l} [P_1]_{i_k j_m} = 1. \tag{23}$$

Since, individual entries are non-negative and ν is a probability vector, this implies $\sum_{m=1}^{l} [P_1]_{i_k j_m} = 1 \ \ k = 1, \ldots, l$

i.e., the row sums are 1. Using formula (9) for the individual matrix entries, this gives

$$m(T^{-1}(\cup_{m=1}^{l} D_{j_m}) \cap D_{i_k}) \quad = \quad m(D_{i_k}) \ \text{ for } k = 1, \ldots, l. \tag{24}$$

where we have used the fact that the pre-image sets are disjoint and $\cup T^{-1}(D_{j_m}) = T^{-1}(\cup D_{j_m})$. However, by construction $S = \cup_{m=1}^{l} D_{j_m}$ and thus

$$m(T^{-1}(S) \cap D_{i_k}) = m(D_{i_k}) \ \text{ for } k = 1, \ldots, l. \tag{25}$$

Taking a summation $\sum_{k=1}^{l}$ on either side gives

$$m(T^{-1}(S) \cap S) = m(S), \tag{26}$$

precisely as claimed in Eq. (19). This completes the proof for the general case.

2. Suppose P_1 is transient. In order to show that A is coarse stable, we proceed by contradiction. Indeed, using definition 7, if A were not coarse stable then there exists an attractor set $B \subset U \subset X_1$ with a sub-partition $\mathscr{S} = \{D_{s_1}, ..., D_{s_l}\}$, $S = \cup_{k=1}^{l} D_{s_k}$ such that $B \subset S \subset U$ and $T(S) \subseteq S$. Since, the set S is left invariant by mapping T,

$$P_{s_k j} = \frac{m(T^{-1}(D_j) \cap D_{s_k})}{m(D_{s_k})} = 0, \tag{27}$$

whenever $D_j \notin \mathscr{S}$. Moreover, because $T : S \to S$,

$$\sum_{j=1}^{l} [P_1]_{s_i s_j} = 1 \qquad i = 1, ..., l, \tag{28}$$

i.e., P_1 is a Markov matrix w.r.t the finite partition $\mathscr{S}$. Form the general theory of Markov matrix [16], there then exists an invariant probability vector ν such that

$$\nu \cdot P_1^n = \nu, \tag{29}$$

TABLE I

CONDITIONS FOR RECURRENCE AND TRANSIENCE

	Linear (A)	Nonlinear (P_0, P_1)
Invariant set	$0 = A \cdot 0$	$\mu = \mu \cdot P_0$
Spectral condition	$\rho(A) < 1$	$\rho(P_1) < 1$
Series-expansion	$A^T \cdot P \cdot A - P = -Q$	$\bar{\mu} = m \cdot (I - P_1)^{-1}$
Linear Program	$--$	$\bar{\mu} \cdot \mathbb{P}_1 < \bar{\mu}$

for all $n > 0$, and P_1 is not transient. ■

In summary, a.e. stability implies P_1 is transient, while one can only conclude a weaker coarse stability given transience of P_1.

C. Formulae for Lyapunov measure

There are a number of equivalent characterizations of the transience, expressed in Definition 6, of the sub-Markov matrix P_1. These are summarized in the theorem below and will be used to obtain computational algorithms for deducing coarse stability.

Theorem 10: Suppose P_1 denotes a sub-Markov matrix. Then the following are equivalent

1) P_1 is transient,
2) $\rho(P_1) \leq \alpha < 1$,
3) the infinite-series $I + P_1 + P_1^2 + \dots$ converges,
4) there exists a Lyapunov measure $\bar{\mu} > 0$ such that $\bar{\mu} P_1 \leq \alpha\bar{\mu}$ where $\alpha < 1$.

Proof: Refer [15] for the proof ■

In summary, transience of the Markov chain P_1 can be expressed in three equivalent ways useful for distinct computational approaches:

1) Verify a spectral condition $\rho(P_1) \leq \alpha < 1$,
2) Compute a Lyapunov measure $\bar{\mu}$ using a series formulation

$$\bar{\mu} = m \cdot (I - P_1)^{-1} = m + m \cdot P_1 + m \cdot P_1^2 + \dots \qquad m > 0$$

3) Compute a Lyapunov measure using a Linear program

$$\bar{\mu} \cdot (\alpha I - P_1) < 0, \quad \bar{\mu} > 0.$$

The parallels with the linear dynamical system are summarized in the Table 1. The spectral condition is a counterpart of $\rho(A) < 1$ for the linear dynamical system. The series expansion corresponds to the series solution of the Lyapunov equation. It can also be obtained as a solution of a linear equation. Finally, the linear program formulation arises due to the non-negativity of the matrix P_1. It does not share any obvious counterpart in the linear setting.

V. EXAMPLES

Example 11: Consider dynamics on a finite set as shown in Fig. 4. The dynamics are defined to be

$$T(x_i) = x_0, \quad \text{for } i = \{1,2\} \; T(y_1) = x_1, \quad \text{for } i = \{1,\dots,N\}. \quad (30)$$

The state $\{x_0\}$ is a globally stable attractor and Table II gives a Lyapunov function and measure on the complement set $\{x_1, y_1, \dots, y_N\}$. The large value of Lyapunov measure $\bar{\mu}$ at the point x_1 is a reflection of the size (N) of its pre-image set. In regions (cells) such as these, where the flow is squeezed through a narrow region, the Lyapunov measure will have a high value. Near the attractor, this property can help one visualize convergence to the attractor set. This is shown with the aid of examples below.

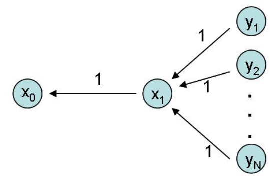

Fig. 4. A schematic for discrete dynamics example with finite number of states.

TABLE II

LYAPUNOV FUNCTION V AND MEASURE $\bar{\mu}$ FOR THE DISCRETE EXAMPLE PROBLEM

Complement set	x_1	y_i
V	$\frac{1}{2}$	1
$\bar{\mu}$	$N+1$	1

Example 12: Consider the 1-d logistic map

$$x_{n+1} = \lambda x_n - x_n^3, \qquad (31)$$

where $\lambda = 2.3$ and $X = [-1.5, 1.5]$ are chosen. The value of λ is specifically chosen to be at the edge of chaos in the logistic map; cf., [12]. Figure 5 (a) shows the asymptotic trajectories as a function of initial conditions in X. There are two symmetric attractors, that a typical initial condition asymptotes too. Figure 5 (b) verifies this fact with the aid of the Lyapunov measure on the complement set to the support of the two invariant measures. We remark that here one does not have global stability for either of the attractors. However, existence of Lyapunov measure on the complement set ensures that in a coarse sense, any initial condition there ends up in the support of one of the two invariant measures.

Example 13: Consider the Vanderpol oscillator given by ODE

$$\ddot{x} - (1 - x^2)\dot{x} + x = 0. \qquad (32)$$

Here, we consider the dynamical system T obtained using numerical integration of the ODE in MATLAB over time-

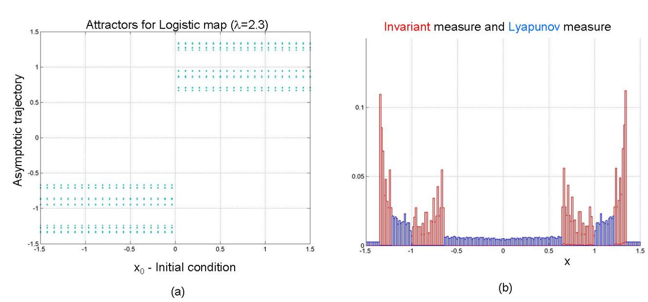

Fig. 5. Asymptotic behavior of the logistic map (a) asymptotic attractor sets as a function of initial condition x_0 and (b) the invariant measures for these attractor sets (in red) and the Lyapunov measure (in blue) showing their stability.

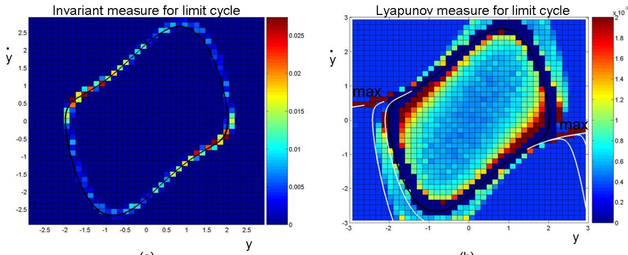

Fig. 6. (a) Invariant measure and (b) Lyapunov measure for the Vanderpol oscillator. The limit cycle is shown as a black curve and white curves in part (b) denote some representative trajectories. For the Lyapunov measure, the maximum value of 0.026 was seen at the two regions denotes as "max." The color axis in part (b) is chopped to $[0, 0.002]$ to better represent the variations in the value of Lyapunov measure.

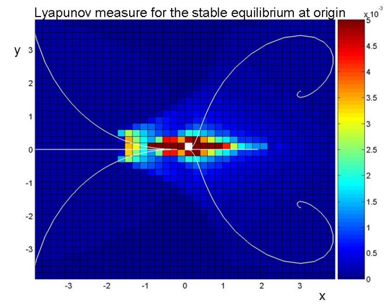

Fig. 7. Lyapunov measure for the stable equilibrium at origin for ODE in Eq. (33) on the glued domain X. The invariant measure is supported on single cell shown in white at the origin. White curves denote some representative trajectories.

interval of 1. The time-interval is chosen to be suitably large so that $T : X \to X$ where the domain $X = [-3,3] \times [-3,3]$ is a finite box containing the unstable origin and the globally stable Vanderpol limit cycle. Figure 6 (a) depicts the approximation of the invariant measure corresponding to this limit cycle and part (b) shows the Lyapunov measure. In the region inside the limit cycle, the measure shows moderate variations with larger values near the limit cycle. Outside the limit cycle, there are two sharp peaks denoting the regions where most trajectories in the phase space are sucked to in before converging uniformly to the vicinity of the limit cycle. The figure shows some of these trajectories (in white) together with the peaks (denoted as "max") in the value of the Lyapunov measure.

Example 14: We next consider the dynamical system T corresponding to the 2D ODE example first given in [3],

$$\dot{x} = -2x + x^2 - y^2, \quad \dot{y} = -6y + 2xy. \tag{33}$$

Unfortunately, this example does not have any finite T-invariant set containing all of the equilibria even with a large value of simulation time-interval. The reason for this is that points on x-axis with $x > 2$ grow unbounded. To overcome this, consider the domain to be $X = [-4,4] \times [-4,4]$ and glue its boundaries. In particular, the left boundary $(x = -4, y)$ is glued to the right boundary at $(x = 4, y)$, the upper boundary $(x, y = 4)$ with $x > 0$ is glued to $(-x, y = 4)$, and similarly on the lower boundary $y = -4$. Inside the glued domain, the dynamics are described by the ODE in Eq. (33). The dynamical system for the same was constructed using numerical integration over time-interval of 0.2. The origin can be verified to be a.e. globally stable for this ODE.

VI. Discussion & Conclusions

Inspired from the Rantzer's result on density function for almost everywhere stability, we propose Lyapunov measure for verifying almost everywhere stability. Set-oriented numerical approaches for the discretization of the stochastic operators is used for the computation of the finite-dimensional approximation of the Lyapunov measure. This finite-dimensional approximation leads to coarser and multi-scale notions of stability which generalizes in a natural way the a.e. stability of [3]. Lyapunov measures can easily be computed using *linear* algorithms. In particular, it is shown that the Lyapunov measure is a solution to a linear program as well as admits a series type expansion.

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