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Controllability for a class of area-preserving twist maps

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Abstract

In this paper, we study controllability of two-dimensional integrable twist maps with bounded area-preserving time-dependent (control) perturbations. In contrast to the time-independent perturbation case of the Kolmogorov–Arnold–Moser theorem, there are no invariant sets other than the whole phase space if the perturbation is made a function of time. We give necessary and sufficient conditions for global controllability of these maps.

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1. Introduction

Control of Hamiltonian systems is a topic that has received a lot of attention lately [1]. Besides the intrinsic beauty of the subject, this is due to a number of exciting applications such as satellite control [2], quantum control [3,4], and control of mixing [5,6].

In this paper, we combine the control-theoretical and dynamical systems point of view to study a class of systems that are well understood from the dynamical systems perspective: perturbations of integrable planar twist maps [7]. These two-dimensional maps defined on an annulus can arise from discretization of continuous-time integrable Hamiltonian systems. Integrable twist maps on an annulus have very simple dynamics given by $(x, y) \to (x + G(y), y)$, with $G'(y) > 0$, where x and y are the usual Cartesian coordinates on the plane and x is considered $\mod 1$. Thus, all the initial conditions stay at the same y for all time and $y =$ constant is an invariant manifold for the dynamics. The Kolmogorov–Arnold–Moser (KAM) theorem [8] (in Moser's version [7]) considers a time-independent perturbation of an integrable twist map. Under the condition that the perturbed map is area-preserving (in fact that it possesses the so-called intersection property, that is implied by area-preservation), KAM theorem states that the majority of initial conditions stay on one-dimensional invariant curves close to the unperturbed invariant curves on which $G(y)$ satisfies the Diophantine condition (strong irrationality). It is commonly

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stated that unperturbed invariant curves that have sufficiently irrational dynamics “persist” under perturbation. The question that we ask here is, how does this change if we allow (bounded) time-dependence of the perturbation? We prove under weak conditions that, for arbitrarily small time-dependent perturbations, every unperturbed invariant curve disappears and global controllability is achieved. This is in marked contrast with the KAM result. To study the controllability of the map we set up the problem in a control-theoretic context.

A question similar to controllability for symplectic maps was asked in [9] by Easton et al. In [10,11], the problem of controlling a Hamiltonian system is considered using small control input. A targeting strategy is suggested which takes advantage of the recurrence property of the system. A similar problem is also addressed in [12]. The problem of targeting in restricted three body problem is studied in [13,14]. By exploiting the natural dynamics of the system a low energy transfer trajectory from earth to moon is constructed. Use of the natural (drift, nominal) dynamics of the system is an important part of the strategy for controllability used here. In fact, the current paper can be seen as a part of the program of investigation of controllability of conservative systems using bounded control, by using ergodic properties of the drift (or uncontrolled) dynamics, started in [15] and described for continuous-time systems in [16]. A combination of Lie-algebraic methods with ergodic properties (such as recurrence) was used already in e.g. [17]. Recently, recurrence properties of the drift were used in [18] while forced recurrence was used in [19].

This paper is organized as follows. In Section 2, we define the problem and provide necessary notation and definitions. The main results follow in Section 3, some consequences and examples in Section 4 and the conclusions are in Section 5.

2. Set-up

In this section, we define two-dimensional map F on the cylinder $\mathcal{A} := S^1 \times \mathbb{R}$, where $S^1 := \mathbb{R}/\mathbb{Z} := [0, 1)$ denotes the circle. The quotient $\mathbb{R}/\mathbb{Z}$ is an equivalence class with $x \equiv y$ if $x - y \in \mathbb{Z}$. A twist map $F : \mathcal{A} \to \mathcal{A}$ is called integrable if it is of the form:

$$F(x, z) = (x + G(z), z), \tag{2.1}$$

where $x \in S^1$, $z \in \mathbb{R}$, and $G'(z) > 0$. For any given $\bar{z}$, the circle of the form $S^1 \times \{\bar{z}\}$ is invariant under the action of the map; i.e., for any $(x, \bar{z}) \in S^1 \times \{\bar{z}\}$, $F^n(x, \bar{z}) \in S^1 \times \{\bar{z}\}\ \forall n \in \mathbb{Z}$.

Two different types of dynamics are possible on each of these invariant circles. For each rational value of $G(\bar{z}) = p/q$, where p and q are integers, the invariant circle $S^1 \times \{\bar{z}\}$ consists of period-q orbits. For all irrational values of $G(\bar{z})$ these invariant circles consist of dense orbits.

Since $G(z)$ is a monotone function, the map F is simplified by defining a new coordinate $y = G(z)$ to obtain the map $S : \mathcal{A} \to \mathcal{A}$ as

$$S(x, y) = (x + y, y).$$

We are interested in studying the dynamics of this map subjected to small time-dependent perturbations. Define a family of discrete time systems $T_{u_t} : \mathcal{A} \times U \to \mathcal{A}$ parameterized by $u_t \in U = [-1, 1]$ and $t \in \mathbb{Z}$ where T_{u_t} is of the following form:

$$T_{u_t}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} x_t + y_t + \epsilon u_t f(x_t, y_t)\ (\mathrm{mod}\ 1) \\ y_t + \epsilon u_t g(x_t, y_t) \end{pmatrix}, \tag{2.2}$$

where $u_t \in U = [-1, 1]$ and $t \in \mathbb{Z}$. We will sometimes call u_t an input, or control input in accordance with control theory literature. We will also sometime call T_{u_t} a perturbed map.

We assume that f and g are at least C^1 (differentiable function with continuous derivative) and periodic in x with period one, i.e. $f(x+1, y) = f(x, y)$, $g(x+1, y) = g(x, y)$. We require that the map (2.2) be area-preserving for any u, i.e., $DT_u|_{(x,y)} = 1$ for any (x, y) and any u. Since f and g are assumed to be C^1, the map $T_u : \mathcal{A} \times U \to \mathcal{A}$ is C^1. We know that $DT_u|_{(x,y)} = 1$ for any (x, y) and any u and thus T_u is a local C^1 diffeomorphism. The composition of maps T_u obtained by applying a sequence of control inputs $u_0, \dots, u_k$ is denoted by

$$T_{u_k,\dots,u_0} := T_{u_k} \circ \cdots \circ T_{u_0}.$$

We study the controllability of (2.2) as in the following definition.

Definition 2.1. The system (2.2) is said to be globally controllable if for any given initial state $(x_0, y_0) \in \mathcal{A}$ and any final state $(x_f, y_f) \in \mathcal{A}$ there exists a sequence of control inputs $u_0, \dots, u_k$ such that $T_{u_k}, \dots, u_0(x_0, y_0) = (x_f, y_f)$.

For the case of time-independent perturbations, say $u_t = 1$ for every t, if the perturbed map satisfies the intersection property—which is guaranteed by area-preservation—and f and g are analytic [7] (or C^r differentiable, $r \geq 4$ [20]) and periodic in x, many invariant curves of the form $y = \phi(x) = \phi(x+1)$ survive. We show that if the perturbation is made a function of time, not only do no invariant curve survive, but global controllability can be obtained. In other words, a sequence of control inputs $\{u_t\}$ exists which can steer the system from any given initial state to any final state.

We need the following definitions.

Definition 2.2. Let $S_k^+(x)$ be the set of points reachable from x in k forward steps and $S^+(x)$ the set of points reachable from x in any positive number of forward steps. Let $S_k^-(x)$ be the set of points controllable to x in k forward steps and $S^-(x)$ be the set of points controllable to x in any positive number of steps.

Definition 2.3. The system is backward accessible from x if the set of points controllable to x (i.e., $S^-(x)$) has a nonempty interior. The system is said to be backward accessible if it is backward accessible from all points.

Definition 2.4. A set $M \subset \mathcal{A}$ is said to be invariant for T if $T_{u_k,\dots,u_0} M \subset M$ for any sequence of control inputs $\{u_0, \dots, u_k\} \in U$.

Note. A set satisfying the above invariance condition is usually called "forward invariant". This is the only notion of invariance we need here—since controllability is a forward-in-time notion. For simpler presentation we keep the current definition.

3. Controllability

Theorem 3.1. *Let f and g in the area-preserving twist map (2.2) be C^1. Let g also satisfy the following regularity condition: there exists a $\delta > 0$ and $\vartheta > 0$ such that*

$$\mu\{x \in S^1 : |g(x, y)| > \vartheta\} > \delta \quad \text{for any fixed } y \in \mathbb{R}, \tag{3.1}$$

where μ is the Lebesgue measure on a line. Then (2.2) is globally controllable if and only if every periodic orbit of the unperturbed map ($\epsilon = 0$) is not invariant for the perturbed map ($\epsilon \neq 0$). The parameter ϵ can be arbitrarily small.

Proof. The necessary part of the proof is obvious: assume that the condition in the theorem is not satisfied, i.e., there exists a periodic orbit P of the unperturbed map which is also invariant for the perturbed map. Consider any point $(x, y) \in P$. Then $T_{u_t}(x, y) \in P$ for any u_t by definition of an invariant set. Hence, from any initial state $(x_0, y_0) \in P$ only states on P can be reached in any number of steps.

To prove the sufficient condition, we will use the following proposition. □

Proposition 3.2. *The area-preserving twist map (2.2) satisfying the regularity condition (3.1) is globally controllable for arbitrary small ϵ if and only if the following conditions are satisfied*:

(1) *For all $y \in \mathbb{Q} \setminus \mathbb{Z}$. If $g(\bar{x}, y) = 0$, then there exists an integer $k_1 \in \mathbb{Z}^+$ such that $g(x_{k_1}, y) \neq 0$,* and *$g(x_k, y) = 0$ for positive $k < k_1$, where $x_k = x_{k-1} + y + \epsilon u_{k-1} f(x_{k-1}, y)$, with $x_0 = \bar{x}$.*

(2) *For all $y \in \mathbb{Z}$*:
   (a) *The functions f and g do not vanish simultaneously*; *i.e.*, *$|f(x, y)| + |g(x, y)| \neq 0$ for all $x \in S^1$.*
   (b) *If $g(\bar{x}, y) = 0$, then there exists an integer $k_2 \in \mathbb{Z}^+$ such that $g(x_{k_2}, y) \neq 0$, and $g(x_k, y) = 0$ for positive $k < k_2$, where $x_k = x_{k-1} + \epsilon u_{k-1} f(x_{k-1}, y)$, with $x_0 = \bar{x}$.*

We prove the sufficient part first. To prove this, we will make use of Lemmas 3.3 and 3.4.

Lemma 3.3. *The area-preserving twist map (2.2) satisfying the regularity condition (3.1) is backward accessible*.

Proof. Consider any point $(x_f, y_f) \in \mathcal{A}$. We have to show that the set of points controllable to (x_f, y_f) contains an open set. To prove this, we show that there exists a sequence of control inputs $\{u_k\}$ such that the inverse image of the map under this sequence of control inputs contains an open set. First we consider the case where y_f is irrational. We know the following:

$$S^{-k}(x', y') = (x' - ky', y')$$

and

$$T_u^{-1}(x', y') = \{(x, y) : x + y + \epsilon uf(x, y) - x' = 0;\ y + \epsilon ug(x, y) - y' = 0\}.$$

Now since y_f is irrational we know that the inverse images of (x_f, y_f) with control inputs zero are dense in $\{y_f\} \times S^1$ and because of the regularity assumption (3.1) we know that there exists an integer k_0 such that $|g(x_f - k_0 y_f, y_f)| > \vartheta$. Now consider the inverse image of (x_f, y_f) under the following sequence of control inputs:

$$T_{u_0}^{-1} \circ \underbrace{S^{-1} \circ \cdots \circ S^{-1}}_{k_0-1}(x_f, y_f) = \{(x, y) : x + y + \epsilon u_0 f(x, y) - x^* = 0;\ y + \epsilon u_0 g(x, y) - y^* = 0\},$$

where $x^* = x_f - (k_0 - 1)y_f$ and $y^* = y_f$. So x and y satisfy

$$x = x^* - y^* - \epsilon u_0 f(x, y) + \epsilon u_0 g(x, y), \qquad y = y^* - \epsilon u_0 g(x, y). \tag{3.2}$$

Since $y^* = y_f$ is irrational, we know that there exist an integer k_1 such that $|g(x^* - k_1 y^*, y^*)| > \vartheta$. Now consider the inverse image of (x, y) under the following sequence of control inputs:

$$\begin{aligned} T_{u_1}^{-1} \circ S^{-(k_1-1)}(x, y) &= T_{u_1}^{-1} \begin{pmatrix} x^* - y^* - \epsilon u_0 f(x, y) + \epsilon u_0 g(x, y) - (k_1 - 1)(y^* - \epsilon u_0 g(x, y)) \\ y^* - \epsilon u_0 g(x, y) \end{pmatrix} \\ &\equiv T_{u_1}^{-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \end{aligned}$$

and

$$T_{u_1}^{-1}(x_1, y_1) = \{(\bar{x}, \bar{y}) : \bar{x} + \bar{y} + \epsilon u_1 f(\bar{x}, \bar{y}) - x_1 = 0; \bar{y} + \epsilon u_1 g(\bar{x}, \bar{y}) - y_1 = 0\}.$$

Substituting the value of x_1 and y_1 from the above expression, we get the following equation to be satisfied by $\bar{x}$ and $\bar{y}$:

$$\begin{aligned} &\bar{x} + \bar{y} + \epsilon u_1 f(\bar{x}, \bar{y}) - x^* + y^* - \epsilon u_0 g(x, y) + \epsilon u_0 f(x, y) + (k_1 - 1)(y^* - \epsilon u_0 g(x, y)) = 0, \\ &\bar{y} + \epsilon u_1 g(\bar{x}, \bar{y}) - y^* + \epsilon u_0 g(x, y) = 0. \end{aligned} \tag{3.3}$$

So $(\bar{x}, \bar{y})$ satisfying Eq. (3.3) with (x, y) satisfying Eq. (3.2) are the set of all points which are mapped to (x_f, y_f) under the following sequence of maps:

$$T_{u_1}^{-1} \circ \underbrace{S^{-1} \circ \cdots \circ S^{-1}}_{k_1 - 1} \circ T_{u_0}^{-1} \circ \underbrace{S^{-1} \circ \cdots \circ S^{-1}}_{k_0 - 1} (x_f, y_f) = (\bar{x}, \bar{y}).$$

Now let $h = (h_1, h_2) : \mathcal{A} \times U \times U \to \mathcal{A}$, where

$$\begin{aligned} &h_1(\bar{x}, \bar{y}, u_0, u_1) = \bar{x} + \bar{y} + \epsilon u_1 f(\bar{x}, \bar{y}) - x^* + y^* - \epsilon u_0 g(x, y) + \epsilon u_0 f(x, y) + (k_1 - 1)(y^* - \epsilon u_0 g(x, y)), \\ &h_2(\bar{x}, \bar{y}, u_0, u_1) = \bar{y} + \epsilon u_1 g(\bar{x}, \bar{y}) - y^* + \epsilon u_0 g(x, y). \end{aligned}$$

Then

$$h(x_f - (k_0 + k_1) y_f, y_f, 0, 0) = 0$$

and

$$\det \left[\frac{\partial h}{\partial(\bar{x}, \bar{y})} \right]_{(x_f - (k_0 + k_1) y_f, y_f, 0, 0)} = \det \begin{bmatrix} \dfrac{\partial h_1}{\partial \bar{x}} & \dfrac{\partial h_1}{\partial \bar{y}} \\ \dfrac{\partial h_2}{\partial \bar{x}} & \dfrac{\partial h_2}{\partial \bar{y}} \end{bmatrix}_{(x_f - (k_0 + k_1) y_f, y_f, 0, 0)} = 1.$$

Hence, by the implicit function theorem, there exists an open neighborhood $\mathcal{O}$ of $(u_0, u_1) = (0, 0)$ and unique functions Ψ_1 and Ψ_2 defined on $\mathcal{O}$ and taking values in $\mathbb{R}$ such that, $\Psi_1(0, 0) = x_f - (k_0 + k_1) y_f$, $\Psi_2(0, 0) = y_f$ and

$$h_1(\Psi_1(u_0, u_1), \Psi_2(u_0, u_1), u_0, u_1) = 0, \tag{3.4}$$

$$h_2(\Psi_1(u_0, u_1), \Psi_2(u_0, u_1), u_0, u_1) = 0 \tag{3.5}$$

for all $(u_0, u_1) \in \mathcal{O}$. Now we have to show that the image of Ψ contains an open set. This is true if $\det[\mathrm{d}\Psi/\mathrm{d}u]_{(0,0)} \neq 0$. We know that in the neighborhood of $(u_0, u_1) = (0, 0)$, we have

$$\begin{aligned} &\left[\frac{\mathrm{d}h}{\mathrm{d}u} \right] = \left[\frac{\partial h}{\partial u} \right] + \left[\frac{\partial h}{\partial \Psi} \right] \left[\frac{\mathrm{d}\Psi}{\mathrm{d}u} \right] = 0, \\ &\det \left[\frac{\mathrm{d}\Psi}{\mathrm{d}u} \right]_{(0,0)} = -\det \left[\frac{\partial h}{\partial \Psi} \right]^{-1}_{(x_f - (k_0 + k_1) y_f, y_f, 0, 0)} \det \left[\frac{\partial h}{\partial u} \right]_{(x_f - (k_0 + k_1) y_f, y_f, 0, 0)}. \end{aligned}$$

We know that $\det[\partial h/\partial \Psi]_{(x_f - (k_0 + k_1) y_f, y_f)} = 1$. So we need to show that $\det[\partial h/\partial u]_{(x_f - (k_0 + k_1) y_f, y_f, 0, 0)} \neq 0$:

$$\det \left[\frac{\partial h}{\partial u} \right]_{(x_f - (k_0 + k_1) y_f, y_f, 0, 0)} = \epsilon^2 (f(\tilde{x}_1, \tilde{y}_1) g(\tilde{x}_2, \tilde{y}_2) - g(\tilde{x}_1, \tilde{y}_1) f(\tilde{x}_2, \tilde{y}_2) - k_1 g(\tilde{x}_2, \tilde{y}_2) g(\tilde{x}_1, \tilde{y}_1)),$$

where $\tilde{x}_1 = x_f - k_0 y_f$, $\tilde{x}_2 = x_f - (k_0 + k_1) y_f$, and $\tilde{y}_1 = \tilde{y}_2 = y_f$. We know that both $g(x_f - k_0 y_f, y_f)$ and $g(x_f - (k_0 + k_1) y_f, y_f)$ are not equal to zero. Now we can make a choice of k_1 which is sufficiently large such that

$\det[\partial h/\partial u]_{(x_f-(k_0+k_1)y_f, y_f, 0, 0)} \neq 0$. The large choice of k_1 is always possible because we know that points iterated under S^{-1} land in the set for which $|g| > \vartheta$ infinitely many times. This proves that the inverse image of (x_f, y_f) contains an open set, when y_f is irrational.

Now we consider the case when y_f is rational. In this case, we only need to show that there exists a sequence of control inputs $\{u_k\}$ such that inverse image of (x_f, y_f) under this sequence contains a point (x_i, y_i) such that y_i is irrational. Once we have proved this, we can show that inverse image contains an open set by using the above argument for irrational y_f.

We know that arbitrary close to y_f there exists an irrational y_i. We write

$$y_f = y_i + \delta_y. \tag{3.6}$$

Since y_i is irrational we know that inverse images of (x_f, y_i) with zero control inputs are dense in $\{y_i\} \times S^1$ and hence there exists an integers k_0 and k_1 such that $|g(x_f - k_0 y_i, y_i)| > \vartheta$ and $|g(x_f - (k_0 + k_1)y_i, y_i)| > \vartheta$. Let $N = k_0 + k_1$ and $x_i = x_f - Ny_i$. We claim that there exists control inputs $u_0^* \in U$ and $u_1^* \in U$ such that

$$S^{k_0-1} \circ T_{u_0^*} \circ S^{k_1-1} \circ T_{u_1^*}(x_i, y_i) = (x_f, y_f). \tag{3.7}$$

This implies that inverse image of (x_f, y_f) contains a point (x_i, y_i), where y_i is irrational. Define a map $\Gamma_{(x_i,y_i)}(u_1, u_0) : U \times U \to \mathcal{A}$ as follows:

$$\Gamma_{(x_i,y_i)}(u_1, u_0) = S^{k_0-1} \circ T_{u_0} \circ S^{k_1-1} \circ T_{u_1}(x_i, y_i).$$

We show that image of the map contains the point (x_f, y_f) and hence there exists u_0^* and u_1^* such that (3.7) is true:

$$\Gamma_{(x_i,y_i)}(u_1, u_0) = \begin{pmatrix} x_f + \epsilon u_1 f(x_i, y_i) + \epsilon u_0 f(x, y) + (N-1)\epsilon u_1 g(x_i, y_i) + (k-1)\epsilon u_0 g(x, y) \\ y_i + \epsilon u_1 g(x_i, y_i) + \epsilon u_0 g(x, y) \end{pmatrix},$$

where $x = x_i + k_1 y_i + \epsilon u_1 f(x_i, y_i) + (k_1 - 1)\epsilon u_1 g(x_i, y_i)$ and $y = y_i + \epsilon u_1 g(x_i, y_i)$. We know that

$$\Gamma_{(x_i,y_i)}(0, 0) = (x_f, y_i),$$

$$\det\left[\frac{\partial \Gamma}{\partial(u_1, u_0)}\right]_{(0,0)} = \epsilon^2 (f(x_i, y_i) g(\tilde{x}, \tilde{y}) - g(x_i, y_i) f(\tilde{x}, \tilde{y}) + k_1 g(\tilde{x}, \tilde{y}) g(x_i, y_i)),$$

where $\tilde{x} = x_f - k_0 y_i$ and $\tilde{y} = y_i$. Since both $|g(\tilde{x}, \tilde{y})| > \vartheta$ and $|g(x_i, y_i)| > \vartheta$, the determinant can be made nonzero by choosing large value of k_1. The large choice of k_1 is always possible to make because point iterated under S^{-1} lands in the set for which $|g| > \vartheta$ infinitely many times. This proves that Γ is a local diffeomorphism at (0, 0) and maps neighborhood of (0, 0) to the neighborhood of (x_f, y_i). The area of the image set mapped under Γ is directly proportional to the determinant of $(\partial\Gamma/\partial(u_1, u_0))$ and can be made bigger by choosing large value of k_1. So by choosing δ_y sufficiently small and k_1 sufficiently large we can ensure that the point (x_f, y_f) belongs to the image of Γ and hence there exists control inputs $u_0^* \in U$ and $u_1^* \in U$ such that $\Gamma_{(x_i,y_i)}(u_1^*, u_0^*) = (x_f, y_f)$. □

The open set of points from which (x_f, y_f) is reachable can without loss of generality be assumed to be an open rectangle $\mathcal{V}$.

Lemma 3.4. *Consider the area-preserving twist map (2.2) satisfying the regularity condition (3.1) and conditions* 1 *and* 2 *of* Proposition 3.2. *Given any initial state* $(x_0, y_0) \in \mathcal{A}$, *there exists a finite sequence of control inputs* $\{u_0, \ldots, u_{\ell-1}\}$ *such that* y_ℓ *is irrational and arbitrarily close to* y_0, *where* $(x_\ell, y_\ell) = T_{u_{\ell-1},\ldots,u_0}(x_0, y_0)$.

Proof. If y_0 is irrational, then $\ell = 0$. If y_0 is rational, then we can consider two different cases: $g(x_0, y_0) = 0$ and $g(x_0, y_0) \neq 0$:

- When $g(x_0, y_0) \neq 0$, then $y_1 = y_0 + \epsilon u_0 g(x_0, y_0)$ can be chosen to be irrational by properly selecting the value of u_0. Since irrational numbers are dense in $[0, 1)$, y_1 can be made arbitrarily close to y_0 by making u_0 sufficiently small. So for this case $\ell = 1$.
- When y_0 is rational and $g(x_0, y_0) = 0$, we can again consider two different cases: noninteger rational $y_0 \in \mathbb{Q} \setminus \mathbb{Z}$ and integer $y_0 \in \mathbb{Z}$.
  - When $y_0 \in \mathbb{Q} \setminus \mathbb{Z}$ and $g(x_0, y_0) = 0$, then there exists an integer $k_1 \in \mathbb{Z}^+$ such that $g(x_{k_1}, y_0) \neq 0$ for $(x_{k_1}, y_0) = T_{u_{k_1-1},\dots,u_0}(x_0, y_0)$. With $g(x_{k_1}, y_{k_1} = y_0) \neq 0$, y_{k_1+1} can be made irrational with proper choice of u_{k_1}, since

    $$y_{k_1+1} = y_{k_1} + \epsilon u_{k_1} g(x_{k_1}, y_0),$$

    where y_{k_1+1} can be made arbitrarily close to y_0 by making u_{k_1} sufficiently small and hence $\ell = k_1 + 1$ for this case.
  - When $y_0 \in \mathbb{Z}$ and $g(x_0, y_0) = 0$, then f and g do not vanish simultaneously and there exists an integer $k_2 \in \mathbb{Z}^+$ such that $g(x_{k_2}, y_0) \neq 0$ for $(x_{k_2}, y_0) = T_{u_{k_2-1},\dots,u_0}(x_0, y_0)$. With $g(x_{k_2}, y_{k_2} = y_0) \neq 0$, y_{k_2+1} can be made irrational with proper choice of u_{k_2}:

    $$y_{k_2+1} = y_{k_2} + \epsilon u_{k_2} g(x_{k_2}, y_0),$$

    where y_{k_2+1} can be made arbitrarily close to $y_0 = y_{k_2}$ by making u_{k_2} sufficiently small. □

The control strategy consists of the following (see Fig. 1): from (x_0, y_0) get to (x_ℓ, y_ℓ) using Lemma 3.4 then turn on the input (either positive or negative depending upon whether $y_\ell < y_f$ or $y_\ell > y_f$ and $g(x, y) > 0$ or $g(x, y) < 0$) whenever $|g(x, y)| > \vartheta$ until $y_k \in \mathcal{V}_y$, where $\mathcal{V}_y$ denotes projection of the open rectangle $\mathcal{V}$ to the y axis. Once $y_k \in \mathcal{V}_y$, input is made zero until $x_k \in \mathcal{V}_x$. With $x_k \in \mathcal{V}_x$ and $y_k \in \mathcal{V}_y$, $(x_k, y_k) \in \mathcal{V} \subset \mathcal{U}$; hence, the system can make the transition to (x_f, y_f) by Lemma 3.3. The detailed proof of this mechanism follows.

Proof of Proposition 3.2. Let (x_0, y_0) and (x_f, y_f) be the initial and final state, respectively. Since the system is backward accessible by Lemma 3.4, the set of points $\mathcal{U}$ controllable to (x_f, y_f) contains an open set; hence there exists an open rectangle $\mathcal{V} \subset \mathcal{U}$. Let $\pi_1(\mathcal{V}) = \mathcal{V}_x$ and $\pi_2(\mathcal{V}) = \mathcal{V}_y$ where π is a projection map and $\pi_i(x_1, x_2) = x_i$ for $i = 1, 2$. We will show that there exists a sequence of inputs $\{u_t\}$ such that $\pi_1(T_{u_k,\dots,u_0}(x_0, y_0)) \in \mathcal{V}_x$ and $\pi_2(T_{u_k,\dots,u_0}(x_0, y_0)) \in \mathcal{V}_y$.

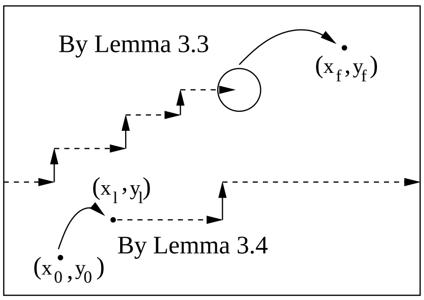

Fig. 1. Use of Lemmas 3.3 and 3.4 in the proof of controllability.

Starting with the initial state (x_0, y_0), we know by Lemma 3.4 that by using a sequence of inputs $u_0, u_1, \dots, u_{\ell-1}$ the system can get to (x_ℓ, y_ℓ) such that y_ℓ is irrational. Let $\bar{y} \in \mathcal{V}_y$ be such that $\bar{y} - y_\ell =: p$ is a rational number and let $m \in \mathbb{Z}^+$ be such that $p/m = \alpha \in (-\epsilon\vartheta, \epsilon\vartheta)$. With y_ℓ irrational the orbit of the rotation map given by

$$x_{t+1} = x_t + y_t \pmod 1 \tag{3.8}$$

is dense in $[0, 1)$ for $y_t = y_\ell$; hence, there exists an integer $k_1 - 1$ such that $|g(x_{\ell+k_1-1}, y_{\ell+k_1-1} = y_\ell)| > \vartheta$ since the set of points $\{x \in S^1 : |g(x, y_\ell)| > \vartheta\}$ is of positive measure by the regularity assumption. With $|g(x_{\ell+k_1-1}, y_\ell)| > \vartheta$ input $u_{\ell+k_1-1}$ can be chosen so that

$$y_{\ell+k_1} = y_{\ell+k_1-1} + \alpha = y_\ell + \alpha.$$

Now $y_{\ell+k_1}$ is still irrational because $y_{\ell+k_1-1}$ is irrational and α is rational. So again the orbit of the rotation map given by

$$x_{t+1} = x_t + y_t \pmod 1$$

is dense in $[0, 1)$; hence, there exists an integer k_2 such that $|g(x_{\ell+k_1+k_2-1}, y_{\ell+k_1+k_2-1} = y_{\ell+k_1})| > \vartheta$ and, with proper choice of $u_{\ell+k_1+k_2-1}$, we have

$$y_{\ell+k_1+k_2} = y_{\ell+k_1+k_2-1} + \alpha = y_{\ell+k_1} + \alpha = y_\ell + 2\alpha,$$

with $y_{\ell+k_1+k_2}$ irrational; the above procedure can be repeated $m - 2$ more times to get

$$y_K = \bar{y},$$

where $K = \ell + \sum_{i=1}^{m} k_i$. Now $y_K = \bar{y} \in \mathcal{V}_y$ is also irrational. With y_K irrational, the orbit of the rotation map given by

$$x_{t+1} = x_t + y_t \pmod 1$$

is dense in $[0, 1)$ and hence there exists an integer n such that $x_{K+n} \in \mathcal{V}_x$. With $x_{K+n} \in \mathcal{V}_x$ and $y_{K+n} = y_K \in \mathcal{V}_y$, $(x_{K+n}, y_{K+n}) \in \mathcal{V} \subset \mathcal{U}$, the system is controllable, since all the points of $\mathcal{U}$ are controllable to $(x_\mathrm{f}, y_\mathrm{f})$.

Now we show that the conditions in the proposition are also necessary for the controllability. Assume that the condition 1 is not satisfied; i.e., there is a point $(\bar{x}, \bar{y})$ such that $\bar{y} \in \mathbb{Q} \setminus \mathbb{Z}$, $g(x_k, \bar{y}) = 0$ for every $x_k = x_{k-1} + \bar{y} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$, with $x_0 = \bar{x}$. Then the set of points reachable from $(\bar{x}, \bar{y})$ is a subset of $y = \bar{y}$.

Assume that condition 2a is not true; i.e., there exists $\tilde{x} \in S^1$ and $\tilde{y} \in \mathbb{Z}$ such that $f(\tilde{x}, \tilde{y}) = g(\tilde{x}, \tilde{y}) = 0$; then

$$x_1 = \tilde{x} + \tilde{y} + \epsilon u_0 f(\tilde{x}, \tilde{y}) \pmod 1 = \tilde{x}, \qquad y_1 = \tilde{y} + \epsilon u_0 g(\tilde{x}, \tilde{y}) = \tilde{y}. \tag{3.9}$$

So $(\tilde{x}, \tilde{y})$ is a fixed point of the map and the system is uncontrollable from $(\tilde{x}, \tilde{y})$.

Assume that condition 2a is true but condition 2b is not true; there is a point $(\bar{x}, \bar{y})$ such that $\bar{y} \in \mathbb{Z}$, $g(x_k, \bar{y}) = 0$ for every $x_k = x_{k-1} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$. Then the set of points reachable from $(\bar{x}, \bar{y})$ is a subset of $y = \bar{y}$. □

Proof of Theorem 3.1. Necessity is already proven. For sufficiency, we need to show that if every periodic orbit of the unperturbed map ($\epsilon = 0$) is not invariant for the perturbed map ($\epsilon \neq 0$) then conditions 1 and 2a–b of Proposition 3.2 are satisfied. These conditions in turn imply controllability.

We start with the easiest, condition 2a. Assume that every periodic orbit of the unperturbed map ($\epsilon = 0$) is not invariant for the perturbed map ($\epsilon \neq 0$). This implies condition 2a, since if for some $y \in \mathbb{Z}$, $x \in S^1$ the functions f and g vanish simultaneously, then that fixed point (or equivalently 0-period orbit) is invariant under the perturbed map and we get a contradiction.

We prove condition 2b by contradiction, i.e., assume that there exists $\bar{y} \in \mathbb{Z}$, $\bar{x} \in S^1$ such that $g(\bar{x}, \bar{y}) = 0$, and there is no integer $k \in \mathbb{Z}^+$ such that $g(x_k, \bar{y}) \neq 0$, where $x_k = x_{k-1} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$, with $x_0 = \bar{x}$. There are two cases:

(1) Consider first the possibility that there is k (and the associated set of inputs $u_0, u_1, \dots, u_{k-1}$) such that $|x_k - \bar{x}| > 1$, where the difference $x_k - x$ is not taken $\bmod 1$. We know that $g(\hat{x}, \bar{y}) > 0$ for some $\hat{x} \in S^1$ by the regularity condition. But since the orbit $x_k = x_{k-1} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$ makes the full circle, there must exist x_b and x_a such that x_a is reached from $\bar{x}$ and $x_b - x_a = \epsilon u_l f(x_a, \bar{y})$ and $\hat{x}$ lies on the arc between x_b and x_a traversed in the direction of the orbit. But then $u_{l1} \in U$ can be chosen so that $\hat{x} = x_a + \epsilon u_{l1} f(x_a, \bar{y})$ giving a contradiction.

(2) The second possibility is that, for any sequence of inputs $x_k = x_{k-1} + \epsilon u_{k-1} f(x_{k-1}, y)$ lies in some bounded arc on S^1. In this case, consider the sequence $\tilde{x}_k = \tilde{x}_{k-1} + \epsilon f(\tilde{x}_{k-1}, \bar{y})$ obtained with constant input $u_k = 1$. It is sufficient to consider the case when f does not change sign on this orbit since if it did change sign on two consecutive points x_a and x_b on the orbit then there would be a point $\hat{x}$ lying on the arc between x_a and x_b traversed in the direction of the orbit such that $f(\hat{x}, \bar{y}) = 0$. $\hat{x}$ could be reached from x_a by some input u_l, by $\hat{x} = x_a + \epsilon u_l f(x_a, \bar{y})$, giving a contradiction. Thus, take f positive without loss of generality. Then $\tilde{x}_k$ is a monotonically increasing bounded sequence and it converges to some $\tilde{x}^*$. But it is clear that by convergence and continuity of f, $f(\tilde{x}^*, \bar{y}) = 0$, and since by assumption $g(\tilde{x}_k, \bar{y}) = 0$, it follows that $g(\tilde{x}^*, \bar{y}) = 0$ and thus $(\tilde{x}^*, \bar{y})$ is a fixed point for the perturbed map—a contradiction.

Thus, we have proven that conditions 2a–b hold. Now we show that condition 1 is also implied by nonpersistence of unperturbed periodic orbits.

Assume there are $\bar{y} = p/q \in \mathbb{Q} \setminus \mathbb{Z}$, $\bar{x} \in S^1$ such that there is no integer $k \in \mathbb{Z}^+$ such that $g(x_k, \bar{y}) \neq 0$, where $x_k = x_{k-1} + \bar{y} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$, with $x_0 = \bar{x}$. Consider the arc A between $\bar{x}$ and $\bar{x} + \bar{y} (\bmod 1)$ traversed in the direction of the orbit. There are again two possibilities: first, there exists a sequence of inputs $u_0, u_1, \dots, u_{k-1}$ such that the whole arc is traversed by the sequence $x_k = x_{k-1} + \bar{y} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$ (note that some of these inputs could be zero). There is a periodic orbit of the unperturbed map such that for some point on that orbit $g \neq 0$. There is also a point of this orbit $\hat{x}$ that is within the arc A. We can find two points of the sequence $\tilde{x}_{k1}, \tilde{x}_{k2}, \tilde{x}_{k2} = \tilde{x}_{k1} + \epsilon u_l f(\tilde{x}_{k1}, \bar{y})$ such that $\hat{x}$ is on the arc between $\tilde{x}_{k1}, \tilde{x}_{k2}$ traversed in the direction of the orbit $\tilde{x}_k$. Then it is possible to change the input u_l to obtain $\hat{x} = \tilde{x}_{k1} + \epsilon u_l f(\tilde{x}_{k1}, \bar{y})$. Accordingly, it is possible by a sequence of inputs to reach the periodic orbit on which there is a point for which $g \neq 0$, yielding a contradiction.

The second case is when the arc A cannot be traversed. In this case, we can emulate the argument given in the second case for the proof of condition 2b and show that there must be a periodic orbit of the unperturbed map on which both f and g vanish. Since f and g vanish on this periodic orbit of the unperturbed map this periodic orbit is also invariant for the perturbed map and hence we obtain a contradiction. To show this, we consider the sequence of inputs $u_k = \mathrm{sgn}(f(x_k, \bar{y}))$, where $x_k = x_{k-1} + \bar{y} + \epsilon u_{k-1} f(x_{k-1}, \bar{y})$. Consider the sequence $\tilde{x}_k = \tilde{x}_{k-1} + \bar{y} + \epsilon u_{k-1} f(\tilde{x}_{k-1}, \bar{y})$ on A obtained using $u_k = \mathrm{sgn}(f(x_k, \bar{y}))$. By monotonicity and boundedness, it converges to some $\tilde{x}^*$, and thus $f(x_{k-1}, \bar{y})$ converges to zero. By continuity of f and the assumption on g, both of them must be zero on the periodic orbit corresponding to $\tilde{x}^*$, yielding a contradiction. □

4. Some consequences and examples

The regularity condition on the perturbation g in the main theorem is not required if the phase space is compact. On an annulus we have the following corollary.

Corollary 4.1. *Let the variable y in the area-preserving twist map (2.2) belong to the closed interval $[a, b]$, f and g be C^1 and for every $y \in [a, b]$ there exists $\tilde{x}(y) \in \mathbf{S}^1$ such that $|g(\tilde{x}(y), y)| \neq 0$. Then, (2.2) is globally controllable if and only if every periodic orbit of the unperturbed map is not invariant for the perturbed map.*

Proof. We only need to show that on the compact domain $D = S^1 \times [a, b]$ perturbation g satisfies the regularity condition, then rest of the proof follows from the proof of the main theorem. Let $x^*(y)$ be the angle at which $\max_x |g(x, y)|$ is reached. This maximum is nonzero. It can be shown that the function $g(x^*(y), y)$ is a continuous function of y (for the proof see [21]). Let

$$g_{\min} = \min_{y \in [a,b]} |g(x^*(y), y)|.$$

This minimum exists and is nonzero due to the fact that $|g(x^*(y), y)|$ is a positive continuous function on a compact domain. Let

$$g_{x\max} = \max_D \left| \frac{\partial g}{\partial x} \right|.$$

We know that for any fixed $y \in [a, b]$ and $x \in S^1$ such that $|x^* - x| < \delta$ (this is a interval of length 2δ), we have

$$|g(x, y)| \geq |g(x^*, y)| - \delta g_{x\max} \geq g_{\min} - \delta g_{x\max}.$$

Now choose δ such that $g_{\min} - \delta g_{x\max} > \vartheta > 0$. So we have following regularity condition:

$$\mu\{x : |g(x, y)| > \vartheta\} > \delta \quad \text{for any fixed } y \in [a, b],$$

where μ is the Lebesgue measure. □

The condition of the main theorem is easily satisfied if f does not change sign.

Corollary 4.2. *Let f in the area-preserving twist map (2.2) satisfy*

$$|f| > \vartheta_1 > 0.$$

Let g also satisfy the regularity condition (2.2). Then (2.2) is globally controllable for arbitrarily small ϵ.

Proof. Obvious, since $|f| > \vartheta_1 > 0$ no periodic orbit of the unperturbed twist map can be invariant for the perturbed map. □

A result with less assumptions on the perturbation can be obtained if we relax the condition that the system be globally controllable to that of approximate global controllability.

Definition 4.3. The system (2.2) is said to be controllable almost everywhere (a.e.) if for almost every (with respect to Lebesgue measure) given initial state (x_0, y_0) and almost every final state (x_f, y_f) there exists a sequence of control inputs $u_0, \ldots, u_k$ such that $T_{u_k,\ldots,u_0}(x_0, y_0) = (x_f, y_f)$.

Corollary 4.4. *Let f and g in the area-preserving twist map (2.2) be C^1. Let g also satisfy the regularity condition (3.1). Then (2.2) is controllable a.e. for arbitrary small ϵ.*

Proof. The proof of this corollary can be easily deduced from the proof of Lemma 3.3 and Proposition 3.2. In particular, we know that (x_f, y_f) with y_f irrational is backward accessible under the regularity condition (3.1) (see proof of Lemma 3.3) and so there is an open set $\mathcal{V}$ from which (x_f, y_f) can be reached by a sequence of inputs. We also know by the proof of Proposition 3.2 that $\mathcal{V}$ can be reached by a sequence of inputs from any (x_l, y_l) with irrational y_l. Thus there is a sequence of inputs that takes any initial point with irrational y coordinate to a chosen final state. We complete the proof by realizing that the set $(S^1 \times I) \times (S^1 \times I)$ (where I is the set of irrationals in $\mathbb{R}$) is of full measure in $(S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$. □

Example 4.5. One of the most studied examples [12] of area-preserving maps is the so-called standard map that in its general form (and our notation allowing for time-dependent inputs) reads

$$T_{u_t}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} x_t + y_t + \epsilon u_t f(x_t) \pmod 1 \\ y_t + \epsilon u_t f(x_t) \end{pmatrix}. \tag{4.1}$$

It is easy to check that (4.1) is area-preserving for any u_t. It does not satisfy conditions of Theorem 3.1 since $f = g$ in this example, and g must vanish for at least one point $\bar{x}$ for say $y = 0$. Thus $(\bar{x}, 0)$ is a fixed point for the perturbed map. However, if the function f is in C^1 and satisfies the regularity condition:

$$\mu\{x : |f(x, y)| > \vartheta\} > \delta \quad \text{for any fixed } y \in \mathbb{R},$$

then it satisfies conditions of Corollary 4.4 and thus the standard map (4.1) is controllable almost everywhere.

We can also show that the map (2.2) has strong controllability properties using only one sequence of control inputs. By this we mean that the system can be steered from arbitrary close to any given initial state to arbitrary close to any final state using only one sequence of control inputs.

Theorem 4.6. *Let f and g in the area-preserving map (2.2) be C^1 and let g also satisfy regularity condition (3.1). Then there is a sequence of control inputs $\{u_k\}$ such that (2.2) has a dense forward orbit. Moreover, for any (x_i, y_i), (x_f, y_f) there is a subsequence of inputs $\{u_{k^*}, u_{k^*+1}, \ldots, u_{k^*+N}\}$ such that there is a trajectory from an arbitrarily small open neighborhood of (x_i, y_i) to an arbitrarily small open neighborhood of (x_f, y_f).*

Proof. Fill $\mathbb{R}$ with nested intervals of the form (a_i, b_i), $i \in \mathbb{Z}^+$ such that $a_i > a_{i+1}$, $b_i < b_{i+1}$ and $\cup_i (a_i, b_i) = \mathbb{R}$. We will construct a sequence of control inputs $\{u_k^i\} = \{u_0^i, \ldots, u_{N_i}^i\}$ which will steer the system from the initial state (x_0, y_0) which is γ_i close to $(0, a_i)$ to the final state which is γ_i close to $(0, a_{i+1})$. The system will be steered using the sequence of control inputs $\{u_k^i\}$ in such a way that the orbit starting from initial state (x_0, y_0) gets γ_i close to every point in the interval (a_{i+1}, b_i). Once this is done, the statement of the theorem easily follows by choosing a sequence $\{\gamma_i\}$ such that $\gamma_i \to 0$ as $i \to \infty$.

Once such a input sequence is constructed it is easy to show that the system is globally approximately controllable with proper initialization of the control input.

We can assume that y_0 is irrational and $|(0, y_0) - (0, a_i)| \leq \gamma_i$, where $|(x_1, y_1) - (x_2, y_2)| = \max(|x_1 - x_2|, |y_1 - y_2|)$ (we say that y_0 is γ_i close to a_i). Starting from (x_0, y_0), set $u_k^i = 0$ for k sufficiently large such that the orbit starting from x_0, $x_k = x_0 + ky_0$, is γ_i close to every point in $\{a_i\} \times S^1$. Since y_0 is irrational this is always true. We also know, because of the regularity condition, that there exists a positive measure set for each $\{y\} \times S^1$ such that $|g(x, y)| > \vartheta$. So select $u_{k_i}^i$ such that $\bar{y} = y_{k+1} = y_0 + \epsilon u_k g(x_k, y_0)$ is γ_i close to y_0 with $\bar{y}$ irrational and $\bar{y} > y_0$.

Now continue repeating the above procedure till the orbit gets γ_i close to every point on $\{b_i\} \times S^1$. Once the orbit gets γ_i close to every point on $\{b_i\} \times S^1$, again repeat the same procedure but this time select input u such that the levels $y_{k+1} < y_k$. Now repeat this new procedure till the orbit gets γ_i close to every point on $\{a_{i+1}\} \times S^1$ and hence $(0, a_{i+1})$.

For the next sequence of inputs the new measure of closeness, γ_{i+1} will be applied. As $\gamma_i \to 0$ as $i \to \infty$, repeating this procedure will bring the orbit starting from (x_0, y_0) arbitrarily close to any point of $S^1 \times \mathbb{R}$.

To prove the last statement of the theorem, select arbitrary (x_i, y_i), (x_f, y_f) and prescribed closeness γ. We know that there exists $j \in \mathbf{Z}^+$ such that $\gamma_j < \gamma$ and $y_i \in (a_j, b_j)$ and $y_f \in (a_j, b_j)$. We also know that there exists $u^j_{n_1} \in \{u^j_k\}$ and $u^j_{n_2} \in \{u^j_k\}$ such that $T_{u^j_{n_1},\dots,u^j_0,\{u^{j-1}_k\},\dots,\{u^0_k\}}(x_0, y_0) = (x_{n_1}, y_{n_1})$ is γ_j close to (x_i, y_i) and $T_{u^j_{n_2},\dots,u^j_0,\{u^{j-1}_k\},\dots,\{u^0_k\}}(x_0, y_0) = (x_{n_2}, y_{n_2})$ is γ_j close to (x_f, y_f). Now if we consider the input sequence starting at $u^j_{n_1+1}$ and start from the initial state (x_{n_1}, y_{n_1}) (which is within the prescribed open neighborhood of (x_i, y_i)), then $T_{u^j_{n_2},\dots,u^j_{n_1+1}}(x_{n_1}, y_{n_1})$ will be within the prescribed open neighborhood (x_f, y_f). □

A simple corollary of the above theorem is that under the designed control input sequence $\{u_k\}$ there can be no invariant sets of positive measure for the family of maps $\{T_{u_k}\}$.

5. Conclusions

We have proved global controllability of a class of discrete-time nonlinear, area-preserving maps using arbitrarily small control inputs. The KAM result holds true for time-independent perturbation of integrable twist maps that we take as our starting point. We proved that when the perturbation is made a function of time, under weak conditions complete controllability is obtained. Of course, the KAM theorem breaks down when dissipation is introduced. But in our setting the perturbation is nondissipative. The only essential difference with the setting of the KAM theorem is that the perturbation to integrable twist maps is made time-dependent.

In the work of [9], inspired by issues in the topic of Arnold diffusion, existence of drifting in action of trajectories in twist maps coupled to a standard map in the "anti-integrable" limit was shown. In our setting, their results would correspond to the situation where U is a discrete set (e.g. $U = \{-1, 1\}$) since the input is obtained from nondegenerate critical points of a function. With such a set of control inputs, they are able to design trajectories that start below any given y_0 and reach any prescribed level y_1. It is an interesting question whether controllability almost everywhere (defined in Section 4) could be achieved using such a discrete set of inputs. It is known that proving global controllability is equivalent to proving ergodicity of the associated random dynamical systems [22]. In this sense, we show that small perturbations of the twist maps are uniquely ergodic.

The control strategy that we pursue stems from [15], where natural dynamics of the system is used to achieve controllability on groups. Given that phase spaces of integrable Hamiltonian systems are foliated by lower-dimensional tori, these methods prove to be quite useful. Generalization to n-degrees of freedom Hamiltonian systems is currently being pursued.

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