Stochastic Positive Real Lemma and Synchronization Over Uncertain Network

Amit Diwadkar, Sambarta Dasgupta and Umesh Vaidya

Abstract—In this paper, we prove the stochastic version of Positive Real Lemma (PRL) to study the stability problem of nonlinear system in Lur’e form with stochastic uncertainty. We consider the mean square stability problem of system in Lur’e form with stochastic parametric uncertainty affecting the linear part of the system dynamics. The results from stochastic PRL are used to study the problem of synchronization in a system of couple Lur’e system with stochastic uncertainty in the interaction. We provide sufficiency condition for synchronization expressed in terms of the second smallest eigenvalue of the coupling nominal (mean) Laplacian and the statistics of link uncertainty in the form of coefficient of dispersion (CoD). Under the assumption that the individual subsystem has identical dynamics, we show that the sufficiency condition is only the function of dynamics of the individual subsystem and mean network characteristics. This makes the sufficiency condition attractive from the point of view of computation for a large size network system. The sufficiency condition specialized for the case of large scale network with first order agent dynamics clearly bring out the interplay of internal agent dynamics, network topology, and uncertainty statistics for the synchronization of the network. Simulation results for network of coupled oscillators with stochastic link uncertainty are presented to verify the developed theoretical framework.

I. INTRODUCTION

The study of network control systems is a topic that has received lots of attention among the research community lately. There is extensive literature on this topic involving both deterministic and stochastic network systems. Among various problems, the problem of characterizing the stability of estimator and controller design for linear time invariant (LTI) network systems in the presence of channel uncertainty is studied in [1], [2]. The similar problem involving nonlinear and linear time varying dynamics is studied in [3], [4], [5], [6]. The results in these papers discover fundamental limitations that arise in the design of stabilizing controller and estimator in the presence of channel uncertainty. Passivity-based tools are used to study the stability problem for deterministic network systems in [7], [8]. Synchronization of interconnected systems from input-output approach has been studied in [9] and shown to have applications in biological networks. These tools provide for systematic procedure for the analysis and synthesis of deterministic network systems.

In this paper, we combine techniques from passivity theory and stochastic systems to provide sufficient condition for the synchronization of uncertain network systems. This is achieved by proving guaranteeing stochastic stability of the error dynamics of these systems. We prove stochastic version of Positive Real Lemma and provide LMI-based verifiable sufficient condition for the mean square exponential stability of stochastic network. One of the important feature of the stochastic extension of Positive Real Lemma is that the uncertainty enters multiplicatively in the system dynamics. Exiting literature on the use of passivity based tools for analysis of stochastic systems assume additive uncertainty models [10], [11]. The sufficient condition are applied to study the synchronization problem in network of Lur’e systems with uncertain linear interactions among the network components. The sufficiency condition for mean square synchronization of the network is posed in terms of a sufficiency condition for mean square stability for a single subsystem of the network with parametric uncertainty and second smallest eigenvalue of the mean network, also known as the Fiedler eigenvalue. The Fiedler eigenvalue is connected in graph theory literature with connectivity of a graph. The sufficiency condition derived here can be solved using standard LMI techniques to study synchronization of the network, analyze effect of uncertainty in links or design network coupling. As the condition is posed in terms of a single subsystem it significantly reduces the computational complexity associated with the verifying the sufficiency condition. This makes our proposed sufficiency condition very attractive for the stability analysis of large scale uncertain network system.

Another interesting results proved in this paper is the dependence of the sufficiency condition on coefficient of dispersion of the network links. The coefficient of dispersion (CoD), defined as a ratio of variance to mean of a random variable indicates the amount of clustering behavior in the random variable. A CoD less than unity indicates patterns of occurrences that are more regular. A CoD greater than unity indicates clusters of occurrences. Thus if the link in the network turns off then it tends to stay off if the uncertainty has high CoD. Some real life networks display this behavior due to heavy tail distributions of uncertainties [12], [13]. The sufficiency condition derived shows that the synchronization of the network can be characterized by the mean square stability of a single subsystem with parametric uncertainty having CoD twice that of the maximum CoD for the uncertain links in the network. Understanding the interplay and dependence of internal dynamics, uncertainty, and interconnection topology for the network synchronization is a challenging problem. Most of the current literature focus on either one of these three elements. The results derived

A. Diwadkar, S. Dasgupta and U. Vaidya are with the Department of Electrical & Computer Engineering, Iowa State University, Ames, IA 50011 dasgupta@iastate.edu, diwadkar.amit@gmail.com, ugvaidya@iastate.edu
in this paper specialize for the case of first order agent dynamics provide a systematic approach for understanding the interplay of all three factors simultaneously. We discuss these results in section III-D.

The rest of the paper is structured as follows: In section II-A we formulate the general problem of stabilization of Lur’è systems with parametric uncertainty and prove the main results on the stochastic variant of Positive Real Lemma. The problem of synchronization is formulated and solved using the stochastic variant of PRL in section III-A. The results on interplay between internal dynamics, uncertainty, and interconnection topology is discussed in section III-D. Simulation results are presented in section IV followed by conclusions in section V.

II. STABILIZATION OF UNCERTAIN LUR’È SYSTEMS

In this section, we first present the problem of stochastic stability of a Lur’è system with parametric uncertainty. The uncertainty is modeled as an independent identically distributed (i.i.d.) random processes. The main result of this section proves the stochastic version of the Positive Real Lemma.

A. Problem Formulation

We consider a Lur’è system, which has parametric uncertainty in the linear system dynamics. The uncertain system dynamics are described as follows:

\[ x_{t+1} = A(\mathcal{Z}(t))x_t - Bu_t \]
\[ y_t = Cx_t \]  

where, \( x \in \mathbb{R}^n \), and \( y \in \mathbb{R}^m \) and \( \phi(y_t, t) \in \mathbb{R}^m \) is a nonlinear function. The state matrix \( A(\mathcal{Z}(t)) \in \mathbb{R}^{n \times n} \) is uncertain. \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{m \times n} \) are the input and output matrices. The uncertainty is characterized by \( \mathcal{Z}(t) = [\zeta_1(t), \ldots, \zeta_M(t)]^T \), where \( \zeta_i(t) \)'s for \( i \in \{1, \ldots, M\} \) are i.i.d. random processes with zero mean and \( \sigma_i^2 \) variance, i.e., \( E[\zeta_i(t)] = 0 \) and \( E[\zeta_i(t)^2] = \sigma_i^2 \). The schematic of the system is depicted in Fig. 1. We make the following assumptions on the nonlinearity \( \phi(y_t, t) \)

**Assumption 1:** The nonlinearity \( \phi(y_t, t) \) is a monotonic non-decreasing function of \( y_t \) such that, \( \phi'(y_t, t)(y_t - D\phi(y_t, t)) > 0 \).

![Fig. 1. Schematic of the system with parametric uncertainty.](image)

The system, described by (1), encompasses a broad class of problems like stabilization under parametric uncertainty, control and observation of Lur’è system over uncertain channel [14], and network synchronization of Lur’è systems over uncertain links. Next, we state and prove a stochastic version of the Positive Real Lemma and successively use the result for network synchronization. The stochastic notion of stability that we use is the mean square exponential stability and is defined as follows:

**Definition 2:** The system described by Eq. (1) is mean square exponentially stable if \( \exists K > 0 \), and \( 0 < \beta < 1 \) such that

\[ E_{\mathcal{Z}} \| x_t \|^2 \leq K\beta^t \| x_0 \|^2, \quad \forall x_0 \in \mathbb{R}^n. \]  

(2)

where, \( x_t \) evolves according to (1).

B. Main Results

The following theorem is the stochastic version of the Positive Real Lemma providing sufficient condition for the mean square stability of the stochastic Lur’è system, described by (1).

**Theorem 3:** Let \( \Sigma = D + D' \) and \( A_T(\mathcal{Z}(t)) = A(\mathcal{Z}(t)) - B\Sigma^{-1}C \). Then the uncertain Lur’è system in (1) is mean square stable if -

1) there exist symmetric positive definite matrices \( P \) and \( R_P \) such that \( \Sigma - B'PB > 0 \) and,

\[ P = E_{\mathcal{Z}(t)} [A_T'(\mathcal{Z}(t))PA_T(\mathcal{Z}(t))] + R_P + C\Sigma^{-1}C \]
\[ + E_{\mathcal{Z}(t)} [A_T'(\mathcal{Z}(t))PB(\Sigma - B'PB)^{-1}B'PA_T(\mathcal{Z}(t))] \]  

(3)

2) there exist symmetric positive definite matrices \( Q \) and \( R_Q \) such that \( \Sigma - CQC' > 0 \) and,

\[ Q = E_{\mathcal{Z}(t)} [A_T(\mathcal{Z}(t))QC'(\Sigma - CQC')^{-1}CQA_T'(\mathcal{Z}(t))] \]
\[ + E_{\mathcal{Z}(t)} [A_T(\mathcal{Z}(t))PB(\Sigma - B'PB)^{-1}B'PA_T(\mathcal{Z}(t))] \]  

(4)

**Proof:** Please refer to the Appendix section for the proof.  

The generalized version of stochastic Positive Real Lemma, as given by Theorem 3, is now specialized to the case of structured uncertainties. In particular, the structured uncertainties are assumed to be of the form \( A(\mathcal{Z}) = A + \sum_{i=1}^M \xi_i A_i \), where \( \{\xi_i\}_{i=1}^M \) are zero mean i.i.d. random variables, the mean value having been incorporated in the deterministic part of the matrix given by \( A \). The state and output equation for uncertain system becomes,

\[ x_{n+1} = \left(A + \sum_{i=1}^M \xi_i A_i\right) x_n - Bu_n \]
\[ y_n = Cx_n \]  

(5)

The matrices \( A_i \), adjoining to the uncertainties, could be pre-determined or could be designed depending on the problem. For instance, the results developed in [14] are for the scenario, where the matrix \( A_i \) is controller gain. The following Lemma simplifies the generalized stochastic PRL to study the mean square stability of system described by (5).
Lemma 4: The system, described in (5), would be mean square exponentially stable if there exists a symmetric matrix $P > 0$, such that $\Sigma - B'PB > 0$ and,

\[
P = A_0'PA_0 + \sum_{i=1}^{M} \sigma_i^2 A_i'PA_i + A_0'PB(\Sigma - B'PB)^{-1}B'PA_0
\]

\[
\sum_{i=1}^{M} \sigma_i^2 A_i'PB(\Sigma - B'PB)^{-1}B'PA_i + R + C' \Sigma^{-1}C
\]

(6)

for some symmetric matrix $R > 0$ and $A_0 := A - B \Sigma^{-1}C$.

Proof: We substitute $\delta(t) = A + \sum_{i=1}^{M} \xi_i A_i$ in the (3) and utilize the fact $\xi_i$'s are zero mean i.i.d. random variables with variance $\sigma_i^2$. We also $A_T(\Sigma) = A + \sum_{i=1}^{M} \xi_i A_i - B \Sigma^{-1}C := A_0 + \sum_{i=1}^{M} \xi_i A_i$. Hence we get,

\[
E_{\mathcal{Z}(t)} \left[A_T'(\Sigma(t))PA_T(\Sigma(t))\right] = A_0'PA_0 + \sum_{i=1}^{M} \sigma_i^2 A_i'PA_i
\]

(7)

Also we get,

\[
E_{\mathcal{Z}(t)} \left[A_T(\Sigma(t))'PB(\Sigma - B'PB)^{-1}B'PA(\Sigma(t))\right] = A_0'PB(\Sigma - B'PB)^{-1}B'PA_0
\]

\[
= \sum_{i=1}^{M} \sigma_i^2 A_i'PB(\Sigma - B'PB)^{-1}B'PA_i
\]

(10)

Combining equations (7) and (8) and substituting in (3) we get the desired result.

Corollary 5: The system, described in (5), would be mean square exponentially stable if there exists a symmetric matrix $Q > 0$, such that $\Sigma - CQC' > 0$ and,

\[
Q = A_0QA_0' + \sum_{i=1}^{M} \sigma_i^2 A_iQA_i' + A_0QC'(\Sigma - CQC')^{-1}CQA_0'
\]

\[
\sum_{i=1}^{M} \sigma_i^2 A_iQC'(\Sigma - CQC')^{-1}CQA_i' + R + B' \Sigma^{-1}B
\]

(11)

for some symmetric matrix $R > 0$ and $A_0 := A - B \Sigma^{-1}C$.

Proof: Corollary 5 follows from Theorem 3, Lemma 4 and duality.

III. SYNCHRONIZATION OF LUR’E SYSTEMS WITH UNCERTAIN LINKS

In this section, we apply the results developed in previous section in analyzing stability of network of Lur’e systems, coupled through uncertain links. We consider a set of linearly coupled systems in Lur’e form. The links, which connect these systems, are uncertain in nature. In the subsequent section we derive sufficiency condition for the stability of network expressed in terms of the statistics of uncertainty and the mean property of the network, in particular the second largest eigenvalue of the interconnection Laplacian. The condition could be used to judge whether the coupled system with uncertainty could retain its stability if the links binding the individual subsystems start to fail. The stability could be achieved if the uncertainties satisfy prescribed bounds.

A. Formulation of Synchronization Problem

We consider a network of inter-connected systems in Lur’e form. The individual subsystems could be described as follows;

\[
S_k := \{ x_k^{i+1}, y_k^i \} = \begin{cases} A_k x_k^i - B \phi(y_k^i, t) \\ C x_k^i \end{cases}, \quad k = 1, \ldots, N
\]

(12)

where, $x_k^i \in \mathbb{R}^n$, and $y_k^i \in \mathbb{R}^m$ are the states and the output of

$k$th subsystem. The $\phi(y_k, n) \in \mathbb{R}^l$ is a nonlinear function. The state matrix $A \in \mathbb{R}^{n \times n}$ is the state matrix for $k$th subsystem. $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$ are the input and output matrices of the $k$th subsystem. The inter-connected system is depicted in Fig. 2. The non-linearity satisfies the following assumption,

Assumption 6: The non-linearity $\phi_k(y_k^i, t) \in \mathbb{R}$ is globally Lipschitz monotonically nondecreasing function and $C^1$ function of $y_k^i \in \mathbb{R}$ that satisfies Assumption 1. Furthermore, it also satisfies the following condition,

\[
\left(\phi^{{\prime}}(y_k^i) - \phi(y_k^i)\right) \phi(y_k^i) > 0,
\]

for any two systems $S_k$ and $S_j$ and some $\Sigma_4 = D_4 + D_4^T > 0$. The aforementioned assumption is essential for the synchronization of the network. Next, we considered the coupled subsystems, whose stability is to be analyzed. We consider the subsystems described by equation (12) are linearly coupled. The coupled system satisfies the following equation,

\[
x_k^{i+1} = A_k x_k^i - B \phi(y_k^i, t) + \sum_{j=1}^{p} a_{kj} G(y_j^i - y_k^i)
\]

(13)

\[
y_k^i = C x_k^i, \quad k = 1, \ldots, N
\]

where, $a_{kj} \in \mathbb{R}$ represent the coupling link between subsystems $S_k$ and $S_j$, $a_{kk} = 0$ and $G \in \mathbb{R}^{n \times m}$.

Remark 7: The coupled system as described by (13) is the most general form of interaction possible between subsystems. The coupling between subsystems could be either in form of output feedback or state feedback. As the output and states of individual subsystems are related linearly so the form of coupling, as described by (13) includes both the output feedback and state feedback.
Next, we define the graph laplacian \( L_G := [l_{ij}] \in \mathbb{R}^{N \times N} \) as following,
\[
l_{ij} := a_{ij}, \quad i \neq j, \quad l_{ii} := -\sum_{j \neq i} a_{ij}, \quad i = 1, \ldots, N. \tag{14}
\]
Next, all the states of the subsystems are combined to create the states of the coupled system. Finally the coupled system can be rewritten as,
\[
\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t - \tilde{B}\tilde{\phi}(\tilde{y}_t) - (L_G \otimes GC)\tilde{x}_t, \quad \tilde{y}_t = \tilde{C}\tilde{x}_t, \tag{15}
\]
where, \( \otimes \) is the Kronecker product, \( L_n \) is an \( n \times n \) Identity matrix and,
\[
\tilde{A} := I_N \otimes A = \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A \end{bmatrix}
\]
We similarly define \( \tilde{B} := I_N \otimes B, \quad \tilde{C} := I_N \otimes C, \quad \tilde{D}_1 := I_N \otimes D_1 \) and \( \tilde{\Sigma} := D_1 + \tilde{D}_1 > 0 \). We also define \( \tilde{x}_t = [(x_1^t)' \ldots (x_N^t)']', \quad \tilde{y}_t = [(y_1^t)' \ldots (y_N^t)']', \quad \tilde{\phi} = [(\phi_1)' \ldots (\phi_N)']' \).

\section*{B. Modeling Uncertain Links}

We are now ready to study the problem of synchronization where the links of the graph are uncertain (i.e. entries of the Laplacian matrix are uncertain). Let \( \mathcal{S} = \{(i,j) | \text{the link (i,j) is uncertain, } i > j\} \)
be the collection of uncertain links in the network. Hence, for links \((i,j) \in \mathcal{S}, \) we have \( a_{ij} = \mu_{ij} + \xi_{ij}, \) where \( \xi_{ij} \) are zero mean i.i.d. random variables with variance \( \sigma_{ij}^2. \) If \((i,j) \notin \mathcal{S} \) when we have \( a_{ij} = \mu_{ij} \) to be purely deterministic. This framework allows us to study synchronization for Lur'e type systems with a deterministic weighted Laplacian as a special case. Let \( \Xi = \{\xi_{ij}\}_{(i,j) \in \mathcal{S}}. \) Then, the uncertain Laplacian \( L_{\xi}(\Xi) \) will be given as,
\[
L_{\xi}(\Xi) = L_d + \sum_{(i,j) \in \mathcal{S}} \xi_{ij}l_{ij} \tag{16}
\]
where \( L_d \) is the deterministic part of the Laplacian replacing \( a_{ij} \) in \( L_G \) with \( \mu_{ij}, \) i.e. \( L_d = [\mu_{ij}] \). We may also write \( L_d = \ell_{ij} f_{ij} \) where \( \ell_{ij} := [\ell_{ij}(1), \ldots, \ell_{ij}(N)]' \) \( \in \mathbb{R}^N \) is a column vector given by
\[
\ell_{ij}(k) = \begin{cases} 0 & \text{if } k \neq i \neq j \\ 1 & \text{if } k = i \\ -1 & \text{if } k = j \end{cases}
\]
We are interested in finding a sufficiency condition involving \( \sigma_{ij}^2 \) for \((i,j) \in \mathcal{S}, \) which would guarantee the mean square exponential synchronization. The coupled network of Lur'e system can be written as,
\[
\tilde{x}_{t+1} = (\tilde{A} - (L_{\xi}(\Xi) \otimes GC))\tilde{x}_t - \tilde{B}\tilde{\phi}(\tilde{y}_t), \quad \tilde{y}_t = \tilde{C}\tilde{x}_t \tag{17}
\]
We would analyze the stochastic synchronization of system, described by (17). We start with following definition of mean square exponential synchronization.

**Definition 8:** The system, described by (17) is mean square exponentially synchronizing if there exists a \( \beta < 1 \) and \( K(\tilde{e}_0) > 0 \) such that,
\[
E_{x}\|x_t' - x_t^i\|^2 \leq K(\tilde{e}_0)\beta^t \|x_0' - x_0^i\|^2, \quad \forall k, t \in [1, N]\tag{18}
\]
where, \( \tilde{e}_0 \) is function of difference \( \|x_t' - x_t^i\|^2 \) for \( i, t \in [1, N] \) and \( K(0) = K \) for some constant \( K. \)

We now apply change of coordinates to decompose the system dynamics on and off the synchronization manifold. The synchronization manifold is given by \( I = [1, \ldots, 1]' \). We show that the dynamics on the synchronization manifold is decoupled from the dynamics off the manifold and is essentially described by the dynamics of the individual system. The dynamics on the synchronization manifold itself could be stable, oscillatory, or complex. Let \( L_d = V_d \Lambda_d V_d' \) where \( V_d \) is an orthonormal set of vectors given by \( V_d = [\frac{1}{\sqrt{N}} U_d], \quad I = [1 \ldots 1]' \) and \( U_d \) is orthonormal set of vectors also orthonormal to \( I. \) Let \( \tilde{z}_t = (V_d' \otimes I_n) \tilde{x}_t. \) Multiplying (17) from the left by \( V_d' \otimes I_n, \) we get
\[
\tilde{z}_{t+1} = (\tilde{A} - (V_d' L_{\xi}(\Xi) V_d \otimes GC)) \tilde{z}_t - \tilde{B}\tilde{\psi}(\tilde{w}_t) \tag{19}
\]
where \( \tilde{w}_t = \tilde{C}z_t, \) and \( \tilde{\psi}_t = (V_d' \otimes I_n) \tilde{\phi} (\tilde{y}_t). \) We can now write
\[
\tilde{z}_t = [x_t' \quad \tilde{z}_t'], \quad \tilde{\psi}_t = [\tilde{\phi}_t' \quad \tilde{\psi}_t'] \tag{20}
\]
Substituting (20) in (19) we get
\[
\tilde{z}_{t+1} = A\tilde{z}_t - B\tilde{\phi}(\tilde{y}_t) \tag{21}
\]
where \( \tilde{w}_t = \tilde{C}z_t, \) \( \tilde{A} := I_{N-1} \otimes A, \) \( \tilde{B} := I_{N-1} \otimes B, \) \( \tilde{C} := I_{N-1} \otimes C, \) and \( \tilde{D}_1 := I_{N-1} \otimes D_1. \) We now show that for the synchronization of system (17), we only need to stabilize \( \tilde{z}_t \) dynamics. The stability of the system with state \( \tilde{z}_t, \) implies the synchronization of the actual coupled system. This feature is exploited to derive sufficiency condition for stochastic synchronization of the coupled system. In the following Lemma we show the connection between the stability of the described by (21) to the synchronization of the system described by (17).

**Lemma 9:** Mean square exponential stability of system described by (21) implies mean square exponential synchronization of the system (17) as given by Definition 8.

**Proof:** To prove this result, we show that second moment of \( \tilde{z}_t \) dynamics is equivalent to the mean square error dynamics of each pair of systems. We then apply stability results to the error dynamics to complete the proof. For the complete proof please refer to the Appendix section of this paper.

In the following subsection we will provide sufficiency conditions for the mean square exponential synchronization
modified using network characteristics, reducing number of variables to \( \frac{n(n+1)}{2} \).

The new sufficient condition is also very insightful as it highlights the role played by the network property, in particular the second largest eigenvalue of the interconnection Laplacian, and the statistics of uncertainty in the sufficient condition. The statistics of uncertainty is captured using the following definition of coefficient of dispersion.

**Definition 11 (Coefficient of Dispersion):** Let \( \xi \in \mathbb{R} \) be a random variable with mean \( \mu > 0 \) and variance \( \sigma^2 > 0 \). Then, the coefficient of dispersion \( \gamma \) is defined as

\[
\gamma := \frac{\sigma^2}{\mu}
\]

To utilize the above definition in subsequent results we make an assumption on the system

**Assumption 12:** For all edges \((i, j)\) in the network, the mean weights assigned are positive, i.e. \( \bar{\mu}_{ij} > 0 \) for all \((i, j)\).

Furthermore, the coefficient of dispersion of each link is given by \( \bar{\gamma}_{ij} = \frac{\sigma^c_{ij}}{\overline{\mu}_{ij}} \), and \( \bar{\gamma} = \max_{(i,j)} \{ \bar{\gamma}_{ij} \} \). This assumption simply states that the network connections are positively enforcing the coupling.

The following theorem provides a sufficient condition for synchronization of the coupled systems based on the stability of a single modified system.

**Theorem 13:** The coupled system (17) is mean square exponentially synchronized if there exists a symmetric positive definite matrix \( \mathcal{P} \in \mathbb{R}^{N \times N} \) such that

\[
P = (A_0 - \lambda_2 G C)' P (A_0 + \lambda_2 G C) + C' \Sigma_1^{-1} C + R
\]

\[
+ (A_0 - \lambda_2 G C)' P B (\hat{\Sigma}_1 - B' P B)^{-1} B' P (A_0 + \lambda_2 G C)
\]

\[
+ (A_0 - \lambda_2 G C)' P B (\hat{\Sigma}_1 - B' P B)^{-1} B' P A_0 + R
\]

\[
(24)
\]

and \( \bar{\Sigma}_1 > 0 \), \( \bar{B}' \bar{B} > 0 \) for some symmetric matrices \( \mathcal{P} > 0 \) and \( \hat{\Sigma}_1 :=\bar{\Sigma}_1 - \bar{B} \bar{A} \bar{B} \). The proof follows from (17), (22), Lemma 9 and Theorem 3.

Proof: The proof follows from (17), (22), Lemma 9 and Theorem 3. The above sufficiency condition is very difficult to verify for large scale networks due to computational complexity associated with solving the Riccati equation. In particular the matrix \( \mathcal{P} \) is of size \((N-1) \times (N-1)\) having \((N-1)^2 + (N-1)\) variables to be determined. The number of variables increases quadratically with change in system dimension or size of network. In the following results, we exploit the identical nature of system dynamics to provide more conservative sufficient condition but with substantially reduced computational efforts. The sufficiency condition is based upon a single representative dynamical system.
Comparing with condition in Theorem 3, equation (25) is the sufficient condition for mean square stability of
\[ x_{t+1} = (A - \xi G)x_t - B\phi(y_t), \quad y_t = Cx_t \] (26)
where \( \xi \) is an i.i.d. random variable with mean \( \mu_c \) and variance \( \sigma_c^2 \). Thus the coefficient of dispersion of \( \xi \) is given by \( \gamma_c = \frac{\sigma_c}{\mu_c} = 2\bar{\gamma} \). Thus the synchronization of the coupled dynamics is guaranteed by the mean square exponential stabilization of an individual system, with parametric uncertainty in the state matrix multiplying the coupling matrix, having coefficient of dispersion twice that of the maximum coefficient of dispersion of the uncertain links of the network.

D. Interplay of internal dynamics, network topology, and uncertainty characteristics

The objective of this section is to understand the interplay of internal dynamics, network topology, and the uncertainty characteristics for the synchronization of large scale network system. To achieve this goal we consider a first order agent dynamics. The first order agent dynamics allows us to simply the condition of main Theorem 13 and bring out explicitly the connection between the internal dynamics, network topology and uncertainty. We consider first order agent dynamics

\[ x_{t+1} = ax_t - \phi(x_t) \] (27)

where \( x_t \in \mathbb{R} \) is a scalar and \( \phi: \mathbb{R} \to \mathbb{R} \) satisfies \( (\phi(x) - \phi(y))(x - y - \frac{\delta}{2}(\phi(x) - \phi(y))) > 0 \) for all \( x, y \in \mathbb{R} \). The above first order dynamics is special case of Eq. (12) with \( B = C = 1 \). The goal is to synchronize \( N \) first order systems over a network with mean Laplacian graph Laplacian \( L_d \) having Fiedler eigenvalue \( \lambda_2 \), and maximum link uncertainty dispersion coefficient \( \bar{\gamma} \). The other parameters for the problem are internal dynamics of the agents captured by parameter \( a \), sector for nonlinearity \( \delta \), and the interconnection gain \( g \) (refer to Eq. (13)) of the Laplacian. We have following theorem.

**Theorem 15:** Consider the problem of synchronization of \( N \) systems of the form (27) over an uncertain network with mean Laplacian \( L_d \) and network interconnection gain \( g \). The sufficient condition for synchronization is given by
1) \( \delta > 1 \), and
2) \( \kappa < \left( \frac{1}{3} - 1 \right)^2 \)

where \( \kappa = (a_0 - \lambda_2 g)^2 + 2\bar{\gamma}\lambda_2 g^2 \) and \( a_0 = a - \frac{1}{\delta} \).

**Proof:** We substitute the parameters for the 1D system from (27) in equation (24) to obtain the sufficiency condition for synchronization of 1D systems. If we suppose the Lyapunov function \( P \) in (24) is denoted by \( q \in \mathbb{R} \), simplifying the equation we get
\[ q > \left( (a_0 - \lambda_2 g)^2 + 2\bar{\gamma}\lambda_2 g^2 \right) \left( \frac{q\delta}{\delta - q} \right) + \frac{1}{\delta} \] (28)
where \( \delta > q \) and \( a_0 = a - \frac{1}{\delta} \). This may be simplified as
\[ 0 > q + \left( \delta(\kappa - 1) - \frac{1}{\delta} \right)q + 1 \] (29)
where \( \kappa = (a_0 - \lambda_2 g)^2 + 2\bar{\gamma}\lambda_2 g^2 \). Equation (29) is quadratic in the Lyapunov function \( q \). We check for feasibility of a solution for this equation when \( \delta > q \). For the value of the quadratic in (29) to be negative we must have two real roots for the equation given by \( q_1^* \) and \( q_2^* \). The quadratic function of \( q \) is negative if and only if \( q_1^* < q < q_2^* \). Furthermore, we also impose the condition that at least \( \delta > q_1^* \) so that at least some roots satisfy the condition \( \delta > q \). Using these conditions we get the system has positive roots only if
\[ \left( \delta(\kappa - 1) - \frac{1}{\delta} \right)^2 - 4 > 0 \] (30)
Simplifying this condition along with the given condition \( \delta > 1 \) and requirement \( \delta > q \) we get the desired result. ■

To understand the interplay of various parameters involved in the problem, we perform simulation results. We identify the feasibility region in \( \lambda_2 - \bar{\gamma} \) space i.e., range of parameter values where the sufficiency conditions from Theorem 15 are satisfied. We vary \( \bar{\gamma} \) between [0,5] and \( \lambda_2 \) between [0,20]. Examining Fig. 3a, which is obtained for fixed values of \( a = 1.25, g = 0.1 \), and \( \delta = 4 \), we notice some interesting interplay between parameter \( \bar{\gamma} \) and \( \lambda_2 \) which will be noted in the following remark.

**Remark 16:** We observe that, there exists a critical value of \( \lambda_2 \), say \( \lambda_2^* \), below which no synchronization occurs (Region I (yellow) in Fig. 3a). Recall that \( \lambda_2 \) is the measure of connectivity of mean network with larger value of \( \lambda_2 \) implies more connected network. Hence, we require minimum degree of connectivity for the synchronization to occur. The parameter values where synchronization is possible, is marked as Region II (blue) in Fig 3. For fixed value of \( \lambda_2 \), an increase in the value of \( \bar{\gamma} \) (i.e., the uncertainty), will cause the system to cease to be in synchronized state. So as expected, high noise level requires higher degree of connectivity of the network. However, interesting interplay between \( \lambda_2 \) and \( \bar{\gamma} \) is observed for large values of \( \lambda_2 \) (Region I in Fig 3). There exists a critical value of \( \lambda_2 \), say \( \lambda_2^* \), above which no synchronization is possible. Thus if there is too much communication between the systems the uncertainty doesn’t remain localized and spreads within the network faster. Furthermore, we see that what ever the graph there is a maximum amount of noise (\( \bar{\gamma_1} \) in Fig 3a) the network can handle for a given system, and there exists an optimal network connectivity for it. Similar simulation results are also performed by varying parameters \( a, g, \) and \( \delta \) to observe the effects of these parameters. Following conclusions are drawn from these parameter variations.

- The increase in internal instability for higher value of \( a \) with require improved network connectivity for synchronization and hence increase in critical value \( \lambda_2^* \). \( \lambda_2^* \) is also proportional to \( a \) indicating high instability in systems interferes with synchronization if there is high level of communication. In Fig. 3b, we show the region of synchronization/desynchronization in \( \lambda_2 - \bar{\gamma} \) space for parameter values of \( a = 1.5, \delta = 4 \), and \( g = 0.1 \).
- Increase in gain \( g \) leads to increase in region where synchronization occurs in \( \lambda_2 \) - \( \bar{\gamma} \) parameter space. \( \lambda_2^* \)
and $\lambda_i^*$ are inversely proportional to the value of $g$ indicating that high gain is detrimental to synchronization of highly connected network but might aid synchronization of sparsely connected networks. In Fig. 3c, we show the region of synchronization/desynchronization in $\lambda_2 - \gamma$ space for parameter values of $a = 1.25, \delta = 4$, and $g = 0.2$.

- The parameter $\delta$ is inversely proportional to sector of nonlinearity i.e., increase in $\delta$ leads to smaller sector of nonlinearity. $\lambda_i^*$ is independent of $\delta$ while $\lambda_i^*$ is directly proportional to $\delta$. Thus we conclude that high level of communication is harmful for synchronization of highly nonlinear systems with large sectors. In Fig. 3d, we show the region of synchronization/desynchronization in $\lambda_2 - \gamma$ space for parameter values of $a = 1.25, \delta = 8$, and $g = 0.1$.

![Fig. 3. Region of synchronization(blue)/desynchronization(yellow) in $\lambda_2 - \gamma$ parameter space a) $a = 1.25, g = 0.1$, and $\delta = 4$; b) $a = 1.5, g = 0.1$, and $\delta = 4$; c) $a = 1.25, g = 0.125$, and $\delta = 4$; d) $a = 1.25, g = 0.1$, and $\delta = 8$](image)

IV. SIMULATION RESULTS

We consider network of coupled oscillator system with linear coupling and stochastic uncertainty in their interactions. The dynamics of the individual oscillator is given by second order differential equation $\ddot{\theta}_i = \kappa \sin \theta_i$. We write the individual oscillator system in Lur'e form as follows

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix} x - \begin{pmatrix} 0 \\ -\kappa \end{pmatrix} \phi(y), \quad y = (1 \ 0)x$$

where we set $\kappa = 1$. The above system is then discretized using a zero order hold. We assume that the nonlinearity and the network interaction change only at discrete intervals and are constant during an interval. We choose the sampling time to be $T = 0.001$ seconds. The phase space dynamics of the discrete time uncoupled oscillator system in shown in Fig. 5. The phase space dynamics consists of two potential wells with periodic motion in each of the well. The oscillators are initialized so that two of the oscillators starts in one potential well and the other two in the second well. We study the synchronization of four coupled oscillators connected over the network as shown in Fig. 4 with output error coupling. We make the links connecting systems $S_1$ to $S_3$, $S_2$ to $S_3$ and $S_2$ to $S_4$ uncertain which are shown as dashed red lines in Fig. 4.

![Fig. 4. Network connectivity of the systems with uncertain edges in red](image)

The mean Laplacian for this network is given by

$$L_d = \begin{pmatrix} 1 + \mu_{13} & -1 & -\mu_{13} & 0 \\ -1 & 1 + \mu_{23} + \mu_{24} & -\mu_{23} & -\mu_{24} \\ -\mu_{13} & -\mu_{23} & \mu_{13} + \mu_{23} & 0 \\ 0 & -\mu_{24} & 0 & \mu_{24} \end{pmatrix}$$

. The uncertainties are modelled as i.i.d. Bernoulli uncertainties where the link connects with probability $p$ and disconnects with probability $1 - p$ for each uncertainty. The mean value for each connection is $\mu_{13} = \mu_{23} = \mu_{24} = p$ while the coefficient of variation $\gamma = 1 - p$. We choose the coupling matrix $G$ as $G = g[1 \ 1]$ where we set $g = 0.002$.

![Fig. 5. a) State space dynamics of uncoupled oscillator; b) $\theta$ dynamics of four oscillators](image)

In Fig. 6, we show the simulation results for two different value of non-erasure probability $p$. We notice that for $p = 0.05$ the oscillators cannot synchronize and some are retained within their initial condition well. The minimum non-erasure probability required for synchronization, as predicted by solving the sufficiency condition is $p^* = 0.6$. This is obtained by solving the Riccati equation using LMI's and semi-definite programming (SDP) techniques. At $p = 0.6$ we see the systems synchronize and are able to pull the oscillators into a common well.
In this paper we study the problem of synchronization of Lur’e systems over an uncertain network. This problem is presented as a special case of the problem of stabilization of Lur’e system with parametric uncertainty. Other special case of this problem include control of Lur’e system over an uncertain network which have been previously studied by the authors. These results are used to obtain some insightful results for the problem of synchronization over uncertain networks. We conclude that mean square exponential synchronization of the coupled dynamics is governed by mean square exponential stability of a specific system with parametric uncertainty in the state matrix multiplying the coupling matrix, and having coefficient of dispersion twice that of the maximum dispersion in the network links. This sufficient condition may be solved as an LMI using the LMI decomposition similar to the Positive Real Lemma. Utilizing identical dynamics of individual subsystems, we show that the sufficiency condition is only a function of individual subsystem dynamics and mean network characteristics. This makes the sufficiency condition attractive from the point of view of computational complexity for large scale networks. Furthermore, studying the sufficiency condition for special case of 1-D systems, we derive important inferences as to the interplay of various parameters, like system dynamics, coupling, mean network connectivity and randomness in interconnection, and their effect on network synchronization. It is shown that in general a very high amount of communication is detrimental to synchronization of the systems as this allows uncertainty to spread faster through the network rather than remain localized. Also, it is observed that a high coupling gain is detrimental for highly interconnected graph. Finally we present simulation results which show the synchronization of oscillators which is an important problem in various fields from power systems to biology.

V. CONCLUSIONS

References

VI. APPENDIX

In the appendix we provide proofs for some of the important results we prove in the paper.

Proof: [Theorem 3] We show the conditions in Theorem 3 are indeed sufficient by constructing an appropriate Lyapunov function that guarantees mean square stability. We will prove the result in Theorem 3 for Case 1. We then use that result to prove Case 2 as the dual of Case 1. First, note that (3) holds if and only if

\[
P = E_{\Xi(t)} \left[ A'(\Xi(t))PA(\Xi(t)) \right] + R_P
\]

+ \( E_{\Xi(t)} \left[ (A'(\Xi(t))PB - C')(\Xi - B'PB)^{-1}(C - B'PA(\Xi(t))) \right] \)

(31)

The equivalence of the two equations (3) and (31) is observed based on [15] (Proposition 12.1.1). Now consider the Lyapunov function \( V(x_t) = x_t^\prime P x_t \). Then, the condition for the system to be mean square stable with Lyapunov function \( V(x_t) \) is given by

\[
E_{\Xi(t)} [V(x_{t+1}) - V(x_t)] = x_t^\prime (E_{\Xi(t)} [A'(\Xi(t))PA(\Xi(t))] - P)x_t
\]

+ \( 2x_t^\prime E_{\Xi(t)} [A'(\Xi(t))BP(\Xi(t)) - \Phi(x_t, t)] + \Phi'(x_t, t)B^\prime P(\Xi(t)x_t) \)

(32)

Substituting from (31) in (32) we get

\[
E_{\Xi(t)} [V(x_{t+1}) - V(x_t)] = -x_t^\prime R x_t + \Phi(x_t, t) \]

(34)

Applying algebraic manipulations as adopted in [16] to (33) and (34), we get,

\[
E_{\Xi(t)} [V(x_{t+1}) - V(x_t)] = -x_t^\prime R x_t - E_{\Xi(t)} [\zeta(x_t)]
\]

From condition of sector nonlinearity as given in Assumption 1 we get \( \Phi(x_t, t) (y_t - D\Phi(x_t, t)) > 0 \), which gives us,

\[
E_{\Xi(t)} [V(x_{t+1}) - V(x_t)] < -x_t^\prime R x_t < 0
\]

This implies mean square exponential stability of \( x_t \) and hence Case 1 is proved. Case 2 is now the dual to Case 1 by a simple argument as shown in [14].

Proof: [Lemma 9] Consider equation (21). We have

\[
\| \hat{z}_t \|^2 = \hat{z}_t^\prime \hat{z}_t = \hat{x}_t^\prime (U_d \otimes I_n) (U_d^\prime \otimes I_n) \hat{x}_t
\]

(35)

We now have

\[
U_dU_d^\prime = V_dV_d^\prime = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} = I_N \frac{1}{N} 1^N
\]

(36)

Substituting (36) in (35) we get

\[
\| \hat{z}_t \|^2 = \hat{x}_t^\prime \left( I_N \frac{1}{N} 1^N \otimes I_n \right) \hat{x}_t
\]

\[
= \hat{x}_t^\prime \left( I_n \frac{1}{\sqrt{N}} 1 \otimes I_n \right) \left( I_n \frac{1}{\sqrt{N}} 1 \otimes I_n \right) \hat{x}_t
\]

\[
= \hat{x}_t^\prime \left( 1 \otimes I_n \right) \hat{x}_t
\]

\[
= \frac{1}{2N} \sum_{i=1, j \neq i, j = 1}^N (x_t - x_j)^{\prime} (x_t - x_j)
\]

(37)

Now, mean square exponential stability of (21) implies there exists \( K > 0 \) and \( 0 < \beta < 1 \) such that

\[
E_{\Xi(t)} \| \hat{z}_t \|^2 \leq K \| \hat{z}_t \|^2,
\]

\[
E_{\Xi(t)} \sum_{i=1, j \neq i, j = 1}^N \sum_{k=1}^N \| x_t - x_j \|^2 \leq K \sum_{i=1, j \neq i, j = 1}^N \sum_{k=1}^N \| x_t - x_j \|^2,
\]

\[
\Rightarrow \sum_{i=1, j \neq i, j = 1}^N \sum_{k=1}^N \| E_{\Xi(t)} \| x_t - x_j \|^2 \leq K \sum_{i=1, j \neq i, j = 1}^N \sum_{k=1}^N \| x_t - x_j \|^2
\]

This gives us

\[
E_{\Xi(t)} \| x_t^k - x_t^\ell \|^2 \leq K \left( 1 + \sum_{i=1, j \neq i, j = 1}^N \sum_{k=1}^N \| x_t^k - x_t^\ell \|^2 \right) \| x_t^k - x_t^\ell \|^2
\]

(38)

Writing \( \tilde{K}(\hat{z}_0) := K \left( \sum_{i=1, j \neq i, j = 1}^N \sum_{k=1}^N \| x_t^k - x_t^\ell \|^2 \right) \| x_t^k - x_t^\ell \|^2 \)

we get for all systems \( S_k \) and \( S_\ell \),

\[
E_{\Xi(t)} \| x_t^k - x_t^\ell \|^2 \leq \tilde{K}(\hat{z}_0) \| x_t^k - x_t^\ell \|^2
\]

Hence the proof.

Proof: [Theorem 13] We know mean square exponential synchronization is guaranteed by conditions in Lemma 10. Consider \( \mathcal{P} = I_{N-1} \otimes P \) where \( P > 0 \) is a symmetric positive definite matrix that satisfies \( \Sigma_1 - B'PB > 0 \). This gives us \( \Sigma_1 - B'PB > 0 \). Using this we write the condition in (23) as follows

\[
I_{N-1} \otimes P > (A_0 - \Lambda_0 \otimes GC)'(I_{N-1} \otimes P)(A_0 - \Lambda_0 \otimes GC)
\]

\[
+ \sum_{\ell_0} \sigma_{\ell_0} A_{\ell_0}^\prime (I_{N-1} \otimes P)A_{\ell_0} + (A_0 - \Lambda_0 \otimes GC)'(I_{N-1} \otimes P)B \cdots
\]

\[
+ \cdots (\Sigma_1 - B'PB)(I_{N-1} \otimes P)B (\Sigma_1 - B'PB)(I_{N-1} \otimes P)B - 1 (B'(I_{N-1} \otimes P)A_{\ell_0} + I_{N-1} \otimes C\Sigma_1^{-1} C
\]

(39)

Since \( A_{\ell_0} = \hat{\ell}_{\ell_0} \otimes GC \) we can write (38) as

\[
I_{N-1} \otimes P > [A_0 - \lambda_0 \otimes GC]'(I_{N-1} \otimes P)[A_0 - \lambda_0 \otimes GC]
\]

\[
+ \sum_{\ell_0} \sigma_{\ell_0} (\hat{\ell}_{\ell_0} \hat{\ell}_{\ell_0}^\prime \otimes GC)'(I_{N-1} \otimes P)(\hat{\ell}_{\ell_0} \hat{\ell}_{\ell_0}^\prime \otimes GC)
\]

\[
+ [A_0 - \lambda_0 \otimes GC]' (I_{N-1} \otimes (PB (\Sigma_1 - B'PB)^{-1} B'P)) [A_0 - \lambda_0 \otimes GC]
\]

\[
+ \sum_{\ell_0} \sigma_{\ell_0} (\hat{\ell}_{\ell_0} \hat{\ell}_{\ell_0}^\prime \otimes GC) \cdots
\]

\[
\cdots (I_{N-1} \otimes (PB (\Sigma_1 - B'PB)^{-1} B'P)) (\hat{\ell}_{\ell_0} \hat{\ell}_{\ell_0}^\prime \otimes GC)
\]

+ \( I_{N-1} \otimes C\Sigma_1^{-1} C
\)

(39)
where \( [A_0 - \lambda_j GC] = (\Lambda_0 - \Lambda_d \otimes GC) \). Inequality (39) can further be simplified using the kronecker product multiplication rule as

\[
\begin{align*}
I_{N-1} \otimes P &> [A_0 - \lambda_j GC]' (I_{N-1} \otimes P) [A_0 - \lambda_j GC] \\
+ 2 \sum_j \sigma^2 \hat{\lambda}_j \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j &\otimes C G' P G C \\
+ [A_0 - \lambda_j GC]' \left( I_{N-1} \otimes (PB (\Sigma_1 - B' PB)^{-1} B') \right) [A_0 - \lambda_j GC] \\
+ 2 \sum_j \sigma^2 \hat{\lambda}_j \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j &\otimes \left( C G' PB (\Sigma_1 - B' PB)^{-1} B' P G C \right) \\
+ I_{N-1} \otimes C G^2 C^{-1} &\leq \gamma \sum_j \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j \hat{\lambda}_j \leq \gamma \Lambda \lambda_d 
\end{align*}
\]

(40)

We know that

\[
\sum_j \sigma^2 \hat{\lambda}_j \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j = \sum_j \gamma \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j \hat{\lambda}_j \leq \gamma \sum_j \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j \hat{\lambda}_j \leq \gamma \Lambda \lambda_d 
\]

(41)

Now, substituting (41) in (40) a sufficient condition for inequality (40) to hold is given by

\[
\begin{align*}
I_{N-1} \otimes P &> [A_0 - \lambda_j GC]' (I_{N-1} \otimes P) [A_0 - \lambda_j GC] \\
+ [A_0 - \lambda_j GC]' \left( I_{N-1} \otimes (PB (\Sigma_1 - B' PB)^{-1} B') \right) [A_0 - \lambda_j GC] \\
+ 2 \gamma \hat{\lambda}_d \otimes \left( C G' (P + PB (\Sigma_1 - B' PB)^{-1} B') G C \right) \\
+ I_{N-1} \otimes C G^2 C^{-1} &\leq \gamma \sum_j \hat{\mu}_j \hat{\ell}_j \hat{\lambda}_j \hat{\lambda}_j \leq \gamma \Lambda \lambda_d 
\end{align*}
\]

(42)

Equation (42) is essentially a block diagonal equation which gives the sufficient condition for mean square synchronization to be

\[
\begin{align*}
P &> (A_0 - \lambda_j GC)' P (A_0 - \lambda_j GC) + 2 \gamma \hat{\lambda}_j C G' P G C + C G^2 C^{-1} \\
+ (A_0 - \lambda_j GC)' PB (\Sigma_1 - B' PB)^{-1} B' P (A_0 - \lambda_j GC) \\
+ 2 \gamma \hat{\lambda}_j C G' PB (\Sigma_1 - B' PB)^{-1} B' P G C
\end{align*}
\]

(43)

for all non-zero eigenvalues \( \lambda_j \) of \( \hat{\Lambda}_d \). Using Schur complement we can equivalently write (43) for a given \( \lambda_j \) and \( \hat{G} = -G \), as an LMI given by

\[
M_1 + \lambda_j M_2 > 0
\]

(44)

where

\[
M_1 = \begin{bmatrix} P - CG^2 C^{-1} & A_0' P & A_0' PB & 0 & 0 \\
PA_0 & P & 0 & 0 & 0 \\
B' PA_0 & 0 & \Sigma_1 - B' PB & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix} 0 & C G' P & C G' PB & \sqrt{2} \gamma C G' P & \sqrt{2} \gamma C G' PB \\
PG C & 0 & 0 & 0 & 0 \\
B' P G C & 0 & 0 & 0 & 0 \\
\sqrt{2} \gamma B' P G C & 0 & 0 & P & 0 \\
\sqrt{2} \gamma B' P G C & 0 & 0 & 0 & \Sigma_1 - B' PB \\
\end{bmatrix}
\]

Matrix \( M_1 \) given in (45) is positive only if individual system dynamics is stable. We do not consider this case as this makes the synchronization problem trivial. Thus \( M_1 \) is indeterminate. Since we require (44) to hold for all non-zero eigenvalues \( \lambda_j \) of the deterministic Laplacian \( L_d \), we conclude that (44) is most vulnerable to the smallest non-zero eigenvalue of \( L_d \) given by \( \lambda_2 \). \( \lambda_2 \) is known as the Fiedler eigenvalue and is associated with graph connectivity. We thus prove the desired result given in (24). \( \blacksquare \)