Proceedings of the
46th IEEE Conference on Decision and Control
New Orleans, LA, USA, Dec. 12-14, 2007

Observability gramian for nonlinear systems

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Abstract— **In this paper, we propose a novel approach for the construction of observability gramian for nonlinear systems. For linear systems observability gramian is obtained as a solution of matrix Lyapunov equation, for nonlinear systems we show that the observability gramian information can be obtained from the solution of Lyapunov measure equation. Lyapunov measure equation is introduced in [1] to provide necessary and sufficient condition for almost every where stability of an invariant set in nonlinear systems. Gramian result for the linear systems forms a special case of the proposed observability gramian approach using transfer operator. For system with output measurement we use set oriented numerical methods are used for the finite dimensional approximation of the gramian.**

I. Introduction

This paper is concerned with the construction of observability gramian for nonlinear system with output measurement. Observability and controllability gramian were introduced in [2] in the context of linear control systems and play a very important role in control theory. One important application of gramian is in the Hankel norm based model reduction of linear control systems. The basic idea behind this model reduction procedure is that the states which are less observable and controllable are not important from the input-output point of view and hence can be removed in the model reduction procedure.

In this paper, we propose linear transfer operators (Perron-Frobenius and Koopman) based approach for the construction of observability gramian for nonlinear systems. For a given dynamical system one can associate a linear transfer operator, while dynamical systems is used to study the evolution of initial conditions or points in the state space, their associated transfer operators are used to propagate sets (measure supported on sets) or densities on the state space. In dynamical systems community there is an increased research interest in the use of the transfer operators to study global transport properties of dynamical systems. In particular transfer operators and their finite dimensional approximation are used for approximate representation, comparison, and visualization of complex dynamics [3], [4], [5], [6], [7]. In our recent work we have proposed the use of transfer operator in particular Perron-Frobenius operator for stability verification, controller design and optimal control of nonlinear control [1], [8], [9], [10] systems.

The work presented in this paper is the continuation of the research theme on the application of transfer operators in nonlinear control. The ultimate goal of the work reported in this paper is to propose a systematic procedure for the gramian based model reduction of nonlinear control systems. Extension of observability and controllability gramian from linear systems to nonlinear systems have been studied in [11], [12], [13] towards the goal of application to model reduction of nonlinear systems. Linear transfer operator based approach provides a natural way of extending the gramian results from linear systems to nonlinear systems. In particular in this paper we show that the information about the observability gramian for nonlinear systems can be obtained from the Lyapunov measure equation just like gramian for linear systems are obtained as a solution of matrix Lyapunov equation. The Lyapunov measure equation is introduced in [1] for stability analysis of nonlinear systems. Lyapunov measure equation is a linear operator equation. Positive solution of Lyapunov measure equation, called as Lyapunov measure, provides necessary and sufficient condition for almost everywhere stability of nonlinear systems. Hence Lyapunov measure equation forms an infinite dimensional linear counterpart of finite dimensional matrix Lyapunov equation. We show that the observability gramian information for nonlinear systems can be obtained from the Lyapunov measure equation and the observability gramian results for the linear systems can be obtained as a special case of the proposed observability gramian using transfer operator. Although observability and controllability gramian can together be used for the model reduction of nonlinear control systems, information obtained from observability gramian can itself be used for building reduced order observers and estimators and for deciding the physical locations of sensors. These are some of the potential applications of the research in this paper.

This paper is organized as follows. In section II, we present some basic preliminaries on transfer operators and stochastic theory of dynamical systems. In section III, we review some key results from [1] on application of Lyapunov measure equation for stability analysis. In section IV, we present the main result of this paper on the construction of observability gramian for nonlinear systems. Set oriented methods for the finite dimensional approximation are discussed in section V followed by conclusion in section VI.

II. Preliminaries

Consider the following discrete time dynamical system $T : X \to X$ with output measurement $G : X \to Y$.

$$x_{n+1} = T(x_n), \qquad y_n = G(x_n) \tag{1}$$

where $x_n \in X \subset \mathbb{R}^n$ and $y_n \in Y \subset \mathbb{R}^m$ are the state and output measurement respectively. X and Y are assumed to be compact and T and G are assumed to be continuous. We

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assume that there exists an invariant set A for the dynamical system T. Let $\mathscr{B}(X)$ denotes the Borel $\sigma-$ algebra on X and $\mathscr{M}(X)$ be the vector space of real value measure. The mapping T is said to be nonsingular with respect to Lebesgue measure m, i.e., if $m(T^{-1}(B)) = 0$ for all set $B \in \mathscr{B}(X)$ such that $m(B) = 0$. We next introduce some notations and preliminaries for the stochastic theory of dynamical systems for more details refer [14], [4].

Definition 1 (Perron-Frobenius operator): The Perron-Frobenius operator (P-F) $\mathbb{P}_T : \mathscr{M}(X) \to \mathscr{M}(X)$ corresponding to the dynamical system $T : X \to X$ is defined as follows

$$\mathbb{P}_T\mu(B) = \int_X \delta_{T(x)}(A)d\mu(x) = \int_X \chi_A d\mu(x) \tag{2}$$

where $\chi_B(x)$ is the indicator function for the set B and $T^{-1}(B)$ is the pre-image set $T^{-1}(B) := \{x \in X : T(x) \in B\}$

Definition 2 (Koopman operator): The Koopman operator $U_T : C^0(X) \to C^0(X)$ corresponding to the dynamical system T is defined as

$$Uf(x) = f(T(x)) \tag{3}$$

Definition 3 (Invariant measure): A measure $0 \neq \mu \in \mathscr{M}(X)$ is said to be T invariant measure if $\mu(B) = \mu(T^{-1}(B))$ for all set $B \in \mathscr{B}(X)$.

Definition 4 (Ergodic measure): A T invariant measure $\mu \neq 0$ with support in X is called an ergodic measure if for T invariant set $A \subset X$ either $\mu(A) = 0$ or $\mu(A) = \mu(X)$ i.e., μ is concentrated in only one invariant subset.

Assumption 5: We assume that the invariant set A has a ergodic measure μ^* supported on it.

For $f \in C^0(X)$ and $\mu \in \mathscr{M}(\mathscr{X})$, define the inner product as

$$\langle f, \mu \rangle_X = \int_X f d\mu(x)$$

With respect to this inner product, the Koopman operator is dual to the P-F operator where the duality is expressed as

$$\langle U_T f, \mu \rangle_X = \int_X U_T f d\mu(x) = \int_X f(x) d\mathbb{P}\mu(x) \langle f, \mathbb{P}_T \mu \rangle_X \tag{4}$$

The evolution of sets or the measure supported on the sets under the system dynamics (1) with output measurement can be defined using the P-F operator as follows

$$\begin{aligned} \mu_{n+1} &= \mu_n \circ T^{-1} =: \mathbb{P}_T \mu_n \\ Y_n &= \int_X G(x) d\mu_n(x) =: \langle G(x), \mu_n \rangle \end{aligned} \tag{5}$$

III. Lyapunov measure and Lyapunov measure equation

In this section we give a brief overview of some of the key results from [1], [8]. In [1], we proved a theorem providing necessary and sufficient condition for almost everywhere uniform stability of invariant set for the dynamical system T. The notion of almost everywhere stability is defined as follows.

Definition 6 (Almost everywhere uniform stability): An invariant set A is said to be almost everywhere unifromly stable with respect to measure m, if for any $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$\sum_N^\infty m(B_n) < \varepsilon$$

for every set $B \subset X \setminus B_\delta$, where U_δ is the δ neighborhood of an invariant set A for any fixed $\delta > 0$.

Throught this paper, we will assume that U_δ is the δ neighborhood of A for any fixed $\delta > 0$. The stability property of an invariant set in definition (6) is stated in terms of the transient behavior of the system on the complement of an invariant set A^c, hence we define sub-stochastic Markov operator as a restriction of the P-F operator on the complement of the invariant set as follows:

$$\mathbb{P}_T^1[\mu](B) := \int_{A^c} \chi_B(T(x)) d\mu(x) \tag{6}$$

for any set $B \in \mathscr{B}(A^c)$ and $\mu \in \mathscr{M}(A^c)$. Similarly the restriction of Koopman operator on the complement of the invariant set can also be defined as

$$(\mathbb{U}_T^1 f)(x) = f(T(x))$$

for all continuous $f : A^c \to \mathbb{R}$. The duality between the restriction of the two linear operators is expressed as in (4). Necessary and sufficient condition for almost everywhere uniform stability of an invariant set A with respect to meausre m were obtained in the form of existence of the positive solution, *Lyapunov measure* $\bar{\mu}$, to the following *Lyapunov measure equation* were stated in the form

$$\mathbb{P}_T^1 \bar{\mu}(B) - \bar{\mu}(B) = -m(B) \tag{7}$$

The precise theorem for stability as proved in [1] is as follows

Theorem 7: An invariant set A for the dynamical system $T : X \to X$ is almost everywhere uniformly stable with respect to Lebesgue measure m if and only if there exists a measure $\bar{\mu}$ which is equivalent to measure m and is finite on $\mathscr{B}(X \setminus U_\delta)$ and satisfies

$$\mathbb{P}_T^1 \bar{\mu}(B) - \bar{\mu}(B) = -m(B)$$

for any set $B \subset X \setminus U_\delta$, where U_δ is the δ neighborhood of the invariant set A.

Almost everywhere uniform stability with respect to Lebesgue measure initial conditions starting from any given set $B \subset X \setminus U_\delta$ can be studied using the following corollary from [1].

Corollary 8: An invariant set A for the dynamical system $T : X \to X$ is almost everywhere uniformly stable with respect to Lebesgue measure initial condition starting from the set $D \subset X \setminus U_\delta$ if and only if there exists a non-negative measure $\bar{\mu}_D$ which is finite on $\mathscr{B}(X \setminus U_\delta)$ and satisfies

$$\mathbb{P}_T^1 \bar{\mu}_D(B) - \bar{\mu}_D(B) = -m_D(B)$$

for any set $B \subset X \setminus U_\delta$, where m_D is the Lebesgue measure supported on set D and U_δ is the δ neighborhood of the invariant set A.

IV. Observability gramian

In this section we prove the main result of this paper on the construction of observability gramian for nonlinear systems. First we show how Lyapunov measure and Lyapunov measure equation can be used to decompose the state space into fast and slow time zones. To do this we need a definition of residence time and is defined as follows

Definition 9: Let B_1, B_2 be any two subset of $X \setminus U_\delta$. The residence time of set B_1 in set B_2 denoted by t_{B_1,B_2} is the amount of time system trajectory starting from set B_1 will spend in set B_2 and is given by following formula

$$t_{B_1,B_2} = \int_{A^c} \sum_{n=0}^{\infty} \chi_{B_2}(T^n(x)) dm_{B_1}(x) \tag{8}$$

where m_{B_1} is the Lebesgue measure supported on set B_1 and χ_{B_2} is the indicator function supported on set B_2.

The summation inside the integral in above equation is well defined and is finite under the assumption that the system T has an invariant set A, which is almost everywhere uniformly stable. For the special case of $B_2 = X \setminus U_\delta =: X_1$, we get t_{B_1,X_1} and this gives us the information about the time it requires for the system trajectory starting from the initial set B_1 to enter the δ neighborhood of the invariant set and hence can be used to characterize faster and slower dynamics of the system. Larger the value of t_{B_1,X_1} compared to t_{B_2,X_1} longer it takes for the system trajectories starting from the initial set B_1 than B_2 to enter the X_1 and hence slower dynamics of B_1 compared to B_2. Hence the residence time can be used to partition the state space into fast and slow time regions. Now we state a theorem which shows how the Lyapunov measure and Lyapunov measure equation can be used to calculate the residence time.

Theorem 10: Let the invariant set A be a.e. uniformly stable for the dynamical system $T : X \to X$ and let $\bar{\mu}_D$ be the solution of following Lyapunov measure equation

$$\mathbb{P}_1 \bar{\mu}_D(B) - \bar{\mu}_D(B) = -m_D(B)$$

where m_D is the Lebesgue measure supported on the set D, we have

$$t_{D,B} = \bar{\mu}_D(B) \tag{9}$$

for any set $D, B \subset X \setminus B_\delta$.

Proof: From definition (9) and the duality between the Koopman and P-F operator in equation (4) along with the result from corollary (8) we have

$$\begin{aligned} t_{D,B} &= \int_{A^c} \sum_{n=0}^{\infty} \chi_B(T^n(x)) dm_D(x) = \sum_{n=0}^{\infty} \left\langle (\mathbb{U}_T^1)^n \chi_B, m_D \right\rangle_{A^c} \\ &= \sum_{n=0}^{\infty} \langle \chi_B, \mathbb{P}_1^n m_D \rangle = \left\langle \chi_B, \sum_{n=0}^{\infty} \mathbb{P}_1^n m_D \right\rangle \\ &= \langle \chi_B, \bar{\mu}_D \rangle = \bar{\mu}_D(B) \end{aligned} \tag{10}$$

■

The construction of observability gramian that we propose in this paper is motivated from the gramian construction in linear systems. For a stable linear system with output measurement

$$x_{n+1} = S x_n \qquad y_n = C x_n \tag{11}$$

where $x_n \in \mathbb{R}^n$ and $y_n \in \mathbb{R}^m$, the observability gramian $\mathbb{O}$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ and is given by

$$\mathbb{O} = \Phi^* \Phi = \sum_{n=0}^{\infty} (CS^n)^T CS^n \tag{12}$$

where $\Phi : \mathbb{R}^n \to \ell_2[0,\infty)$ and $\Phi^* : \ell_2[0,\infty) \to \mathbb{R}^n$ and Φ^* is adjoint to Φ. The amount of energy in the output starting from any initial state $x_0 \in \mathbb{R}^n$ can now be obtained using the observability gramian and is given as follows.

$$\| y \|^2 = x_0^T \mathbb{O} x_0 \tag{13}$$

Note that the observability gramian can be obtained as a solution of following matrix Lyapunov equation

$$S^T \mathbb{O} S - \mathbb{O} = -C^T C$$

The information about the relative degree of observability of different states in the state space can be obtained from the eigenvalues and eigenvectors of the observability gramian, in particular the states corresponding to eigenvectors with larger eigenvalues are more observable compared to the states with eigenvector with smaller eigenvalues. For more details on observability gramian and its construction for linear systems refer to [15].

Unlike linear systems where observability gramian is used to characterize the degree of observability of the states in the phase space, in our proposed approach we characterize degree of observability of a set in the phase space. Consider a discrete time dynamical system with output measurement

$$x_{n+1} = T(x_n) \qquad y_n = G(x_n) \tag{14}$$

We assume that there exists an invariant set A with ergodic measure μ^* supported on the set. Furthermore set A is assumed to be almost everywhere uniformly stable as per definition (6). The evolution of the system in measure space is described by following equation

$$\begin{aligned} \mu_{n+1}(B) &= \mathbb{P}_T \mu_n(B) \\ Y_n^i &= \textstyle\int_X g_i(x) d\mu_n(x) = \langle g_i(x), \mu_n \rangle \; i = 1,\dots,m \end{aligned} \tag{15}$$

where $G = (g_1, \dots, g_m)^T$ and set $B \in \mathscr{B}(X)$. Since the invariant set A is assumed to be almost everywhere uniformly stable we know from theorem (7), that there exists a finite Lyapunov measure $\bar{\mu}$ on $X \setminus U_\delta$ which solves the Lyapunov measure equation (7). We now make following assumption regarding the integrability of the output measurement G with respect to Lyapunov measure $\bar{\mu}$.

Assumption 11: Assume that each of the function g_i^2 for $i = 1, \dots, m$ is integrable with respect to Lyapunov measure $\bar{\mu}$ on A^c i.e.,

$$\int_{A^c} g_i^2(x) d\bar{\mu}(x) \leq K < \infty \quad i = 1, \dots, m. \tag{16}$$

We now define a observability map Ψ as a mapping from all sets $B \subset X \setminus U_\delta$ to the sequence of real numbers $\Psi : \mathscr{B}(X \setminus U_\delta) \to \ell_1[0,\infty)$ as follows:

$$B \to \begin{cases} O_n := \langle f(T^n(x)), m_B \rangle_{A^c} & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

where $f(x) := (G^T G)(x)$ and m_B is the Lebesgue measure supported on the set B. Note that $O_n \geq 0$ and $O_n \in \ell_1[0,\infty)$ (for proof refer to theorem (13)). The degree of observability of a set B is now defined using the observability map as follows.

Definition 12 (Degree of Observability): The degree of observability for the any set $B \subset X \setminus U_\delta$ is a mapping $\mathscr{O} : \mathscr{B}(X) \to \mathbb{R}$ and is defined as follows

$$\mathscr{O}(B) = \sum_{n=0}^{\infty} O_n \quad \text{for} \quad B \in \mathscr{B}(X \setminus U_\delta) \quad (18)$$

$$\mathscr{O}(B) = \int_B f(x) d\mu^*(x) \quad \text{for} \quad B \in \mathscr{B}(A)) \quad (19)$$

where $f(x) = (G^T G)(x)$ and μ^* is the ergodic measure supported on the invariant set A.

In the next theorem we show that the observability map is well defined i.e., $O_n \in \ell_1[0,\infty)$ and that Lyapunov measure $\bar{\mu}$ can be used to characterize the degree of observability for a set B as defined above.

Theorem 13: Let the invariant set A be almost everywhere uniformly stable for the dynamical system $T : X \to X$ with output measurement $G : X \to Y \subset \mathbb{R}^m$ satisfying the assumption (11). The degree of observability for any set $B \subset X \setminus U_\delta$ is given by

$$\mathscr{O}(B) = \langle f(x), \bar{\mu}_B \rangle \quad (20)$$

where $f(x) := (G^T G)(x)$ and $\bar{\mu}_B$ is the solution of the following Lyapunov measure equation (21).

$$\mathbb{P}_1 \bar{\mu}_B - \bar{\mu}_B = -m_B \quad (21)$$

where m_B is the Lebesgue measure supported on set B. For the special case of stable linear dynamical system $x_{n+1} = Sx_n$ with output $y_n = Cx_n$, the expression (20) reduces to

$$\mathscr{O}(B) = \int_{A^c} x^T P x dm_B(x) \quad (22)$$

where P is the solution of Lyapunov equation

$$S^T P S - P = -C^T C$$

Proof: Using the duality relation (4) between the P-F and the Koopman operator and the definition of observability map (17) we get

$$\begin{aligned} O_n &= \int_{A^c} f(T^n(x)) dm_B(x) = \int_{A^c} f(y) dm_B(T^{-n}(y)) \\ &= \int_{A^c} f(y) dm_B(T^{-n}(y) \cap A^c) = \langle f, \mathbb{P}_1^n m_B \rangle_{A^c} \quad (23) \end{aligned}$$

Now using the result from the corollary (8), we get

$$\mathscr{O}(B) = \sum_{n=0}^{\infty} O_n = \sum_{n=0}^{\infty} \langle f(x), \mathbb{P}_1^n m_B \rangle_{A^c} = \langle f(x), \bar{\mu}_B \rangle_{A^c} \quad (24)$$

From the almost everywhere stability assumption of the invariant set A and from assumption (11), we have

$$\langle f(x), \bar{\mu}_B \rangle_{A^c} \leq K < \infty$$

and hence $O_n \in \ell_1[0,\infty)$.

For the special case of linear system $f(x) = x^T C^T C x$, we have

$$\begin{aligned} \mathscr{O}(B) &= \sum_{n=0}^{\infty} O_n = \sum_{n=0}^{\infty} \langle U_S^n f, m_B \rangle_{A^c} \\ &= \int_{A^c} \sum_{n=0}^{\infty} U_S^n f dm_B(x) = \int_{A^c} \sum_{n=0}^{\infty} x^T (S^T)^n C^T C S^n x dm_B(x) \end{aligned}$$

where the last equality is know to be equal to $\int_{A^c} x^T P x dm_B(x)$ where P is the solution of the following Lyapunov equation.

$$S^T P S - P = -C^T C$$

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For linear system, use of observability gramian in measure space to characterize the degree of observability of a set leads to

$$\mathscr{O}(B) = \int_{A^c} x^T P x dm_B(x)$$

Hence for the special case of Dirac-delta measure supported at x_0, we get the familiar result for linear systems on characterizing the degree of observability of a state i.e.,

$$\mathscr{O}(x_0) = \int_{A^c} x^T P x \delta_{x_0}(x) = x_0^T P x_0$$

In characterizing the degree of observability for a set $B \subset X \setminus U_\delta$ we made use of two pieces of information. First the average residence time of set B in the complement of the invariant set, which is characterize in terms of measure $\bar{\mu}_B$ and the second is the value that the output observable G takes on the set B. Intuition behind characterizing the degree of observability using $\bar{\mu}_B$ and observable G is as follows: Consider the case where the output $G = I$, then the degree of observability of any set B is directly proportional to the resident time of set B in $X \setminus U_\delta$. Longer it takes for the system trajectories starting from the set B to enter the δ neighborhood of an invariant set more observable the set B will be. On the other hand if the residence time of the two different sets in $X \setminus U_\delta$ is same then the degree of observability of these sets is directly proportional to the value function $G^T G$ takes along the trajectories starting from the two sets. In the next section we obtain finite dimensional approximation for the degree of observability for any set B using set oriented numerical methods.

V. Finite dimensional approximation

In this section use set oriented numerical method for the finite dimensional approximation of the gramian. The first step towards this goal is to obtain the finite dimensional approximation of the Perron-Frobenius operator. We provide a brief overview of the finite dimensional approximation of P-F operator for more detail refer to [3], [8]. We consider the finite dimensional approximation of the phase space X, denoted as

$$\mathscr{X} := \{D_1, \cdots, D_L\}, \tag{25}$$

Approximation of the measure $\mu \in \mathscr{M}(X)$ on the finite partition of the state space X is given by

$$d\mu(x) = \sum_{i=1}^{L} \mu_i \kappa_i(x) \frac{dm(x)}{m(D_i)}$$

Hence the infinite dimensional vector space of measure is identified with the finite dimensional vector space $\mu = (\mu_1, ..., \mu_L) \in \mathbb{R}^L$ and the finite dimensional approximation of P-F is identified with a matrix on $\mathbb{R}^L$. The finite dimensional approximation of the P-F is given by

$$P_{ij} = \frac{m(T^{-1}(D_j) \cap D_i)}{m(D_i)}, \tag{26}$$

m being the Lebesgue measure. The resulting matrix is non-negative and because $T : D_i \to X, \sum_{j=1}^{L} P_{ij} = 1$, i.e., P is a Markov or a row-stochastic matrix. Computationally, several short term trajectories are used to compute the individual entries P_{ij}. The mapping T is used to *transport* M "initial conditions" chosen to be uniformly distributed within a set D_i. The entry P_{ij} is then approximated by the fraction of initial conditions that are in the box D_j after one iterate of the mapping. The finite dimensional approximation of P-F operator can be used to obtain the finite dimensional approximation of the Lyapunov measure equation and Lyapunov measure. To do this we first need the restriction of P on the complement of invariant set A^c. Let μ_0 be the finite dimensional approximation of the ergodic measure μ^* and is obtained as the left eigenvector with eigenvalue one of the Markov matrix P i.e.,

$$\mu_0 P = 1.\mu_0$$

We form the partition of the invariant set A and its complement A^c as follows

$$\mathscr{X}_1 = \{D_1, ..., D_K\} \qquad \mathscr{X}_0 = \{D_{K+1}, ..., D_L\}$$

with $A \subset X_0 = \cup_{i=K+1}^{L} D_i$ and $X_1 = \cup_{i=1}^{K} D_i$. The partitions $\mathscr{X}_0$ and $\mathscr{X}_1$ can be associated with measure space $M_0 \cong \mathbb{R}^{L-K}$ and $M_1 \cong \mathbb{R}^K$ respectively and with respect to this splitting the Markov matrix P admits a decomposition into a Markov matrix $P_0 : M_0 \to M_0$ and sub-Markov matrix $P_1 : M_1 \to M_1$. For more details on the finite dimensional decomposition of the P-F operator refer to [8].

The following theorem from [8] guarantee the transient nature of sub Markov matrix P_1 under the assumption that the invariant set A is almost everywhere stable. P_1 is defined to be transient if $\lim n \to \infty [P_1^n]_{ij} = 0$ for $i, j = 1, ..., K$.

Theorem 14: Let the invariant set $A \subset X_0 \subset X$ be almost everywhere uniformly stable (definition (6)) with approximate invariant measure supported on finite partition $\mathscr{X}_0$ of X_0. P_1 is its finite-dimensional sub-Markov matrix approximation obtained with respect to the partition $\mathscr{X}_1$ of the complement set $X_1 = X \setminus X_0$. For this

1) Suppose there exists a measure $\bar{\mu}$ such that

$$\mathbb{P}_T^1 \bar{\mu}(B) - \bar{\mu}(B) = -m(B) \tag{27}$$

for all $B \subset \mathscr{B}(X_1)$, and additionally $\bar{\mu}$ is equivalent to the Lebesgue measure m. Then the finite-dimensional approximation P_1 is transient.

2) Suppose P_1 is transient then A is coarse stable with respect to the initial conditions in X_1.

Proof: Refer to [8] for the proof. ■

For more details on the coarse stability refer to [8]. Since P_1 is transient it implies that $\rho(P_1) \le \alpha < 1$ and hence the infinite series $I + P_1 + P_1^2 + P_1^3 + ...$ converges to $(I - P_1)^{-1}$ and $[(I - P_1)^{-1}]_{ij}$ is finite for $i, j = 1, ..., K$. The sub Markov matrix P_1 can now be used for approximating the average residence time of the set D_i to D_j and from D_i to complement of the invariant set X_1 as follows.

Lemma 15: Let $[(I - P_1)^{-1}]_{ij} = \bar{P}_{ij}$, for $i, j = 1, ..., K$ then for any set $D_i, D_j \subset X_1$, t_{D_i,D_j} is approximately equal to $\bar{P}_{ij}$ and the average residence time of set $D_i \subset X_1$ in the complement of the invariant set X_1 i.e., t_{D_i,X_1} is approximately equal to $\sum_{j=1}^{K} \bar{P}_{ij}$

Proof: From equation (10), we have

$$t_{D_i,D_j} = \left\langle \chi_{D_j}, \sum_{n=0}^{\infty} \mathbb{P}_1^n m_{D_i} \right\rangle$$

Using the finite dimensional approximation of the $P - F$ operator on the finite dimensional measure space $\mathbb{R}^L$, we can approximate t_{D_i,D_j} as follows

$$t_{D_i,D_j} \approx (\mu_i^T \sum_{n=0}^{\infty} P_1^n)(\vartheta_j) = (\mu_i^T (I - P_1)^{-1})(\vartheta_j) \tag{28}$$

where $\vartheta_j, \mu_i \in \mathbb{R}^k$ are assumed to be a column vectors and are finite dimensional approximation of the indicator function supported on set D_j and the Lebesgue measure supported on the set D_i respectively. Vectors ϑ_j and μ_i consists of all zeros except for ones at j^{th} and i^{th} location respectively. After substitution, we get

$$t_{D_i,D_j} \approx \bar{P}_{ij}$$

For the residence time of set D_i in the complement of the invariant set we have

$$t_{D_i,X_1} = \sum_{j=1}^{K} t_{D_i,D_j} \approx \sum_{j=1}^{K} \bar{P}_{ij}$$

■

Theorem 16: The degree of observability of any set $D_i \subset X_1$ for $i = 1, ..., K$ is approximated given as

$$\mathscr{O}(D_i) \approx \sum_{j=1}^{K} \bar{P}_{ij} \alpha_j \tag{29}$$

and the degree of observability for any set $D_i \subset X_0$ for $i = K + 1, ..., L$

$$\mathscr{O}(D_i) \approx \alpha_i \mu_{0_i} \tag{30}$$

where $\alpha_j = \int_{D_j}(G^T G)(x)dm(x)$ and μ_0 is the finite dimensional approximation of the invariant measure μ^* supported on the invariant set A such that $\mu_{0_i} := \mu_0(D_i)$.

Proof: From equation (20), for any set $D_i \subset X_1$ we have

$$\mathscr{O}(D_i) = \int_X f(x)d\bar{\mu}_{D_i}(x) \approx \sum_{j=1}^{K}\int_{D_j} f(x)dm(x)\bar{\mu}_{D_i}(D_j) = \sum_{j=1}^{K}\bar{P}_{ij}\alpha_j \quad (31)$$

where $\alpha_j = \int_{D_j} G^T G(x)dm(x)$. Similarly the degree of observability for any set $D_i \subset X_0$ can be approximated as

$$\mathscr{O}(D_i) = \int_{D_i} f(x)d\mu^*(x) \approx \int_{D_i} f(x)dm(x)\mu_{0_i} = \alpha_i\mu_{0_i}$$

■

A. Example

Example 17: Consider the ODE for the Vanderpol oscillator with scalar output measurement.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + (1-x_1^2)x_2\\ y &= x_1 \end{aligned} \quad (32)$$

The Vanderpol oscillator has a unstable equilibrium point at the origin and a stable limit cycle. The limit cycle for this example is almost everywhere uniformly stable. Figure (1) show the plot for the decomposition of the phase space into fast slow dynamics. Color code indicates relative amount of time trajectories starting from each box spends in the complement of the limit cycle. Larger the value of the color code on the box more time the system trajectory starting from that box spends in the complement of the limit cycle. Figure (2) shows the plot for the finite dimensional approximation of the relative degree of the observability of different sets as defined by the formula (29) for $y = x_1$.

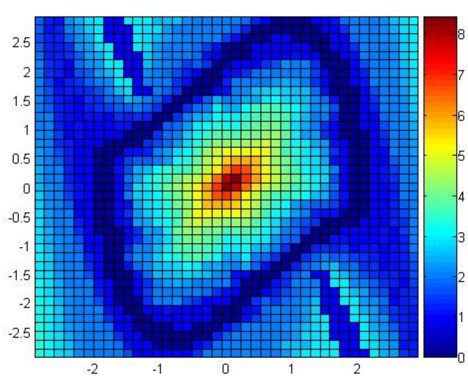

Fig. 1. Decomposition of phase space into fast and slow dynamics

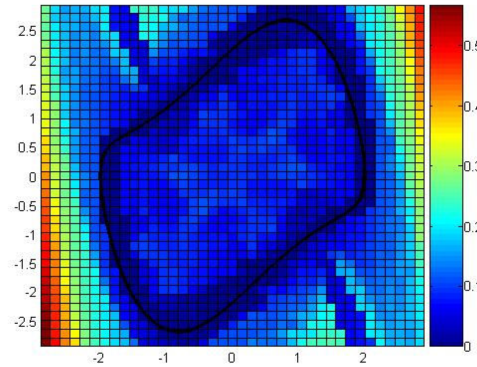

Fig. 2. Approximate observability gramian for $y = x_1$

VI. Conclusions

Using linear transfer operator approach from stochastic theory of dynamical system, we have extended the notion of observability gramian from linear system to nonlinear systems. This extended notion of observability gramian captures the degree of observability of a sets in the phase space. Lyapunov measure is used to partition the phase space into slow and fast time region which along with the output measurement is used in the construction of the gramian.

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