CprE 310 Theoretical Foundations of Computer Engineering

Homework 2

Solution Key

Problem 2.1

Following the convention in the book, $N = \{0, 1, 2, 3 \ldots\}$.

a. Define function f from N to N as $f(n) = n + 1$ for every n.

   f is one-one because $f(n_1) = f(n_2)$ implies $n_1 = n_2$. To see that f is not onto, we observe that there is no $x \in N$ such that $f(x) = 0$.

b. Define function g from N to N as follows.

   $g(0) = 0$, and $g(n) = n - 1$ if $n > 0$.

   g is not one-one because $g(0) = g(1) = 0$.

   However, $g(n + 1) = n$, for every $n \in N$. Thus, for every $x \in N$ there is a number $y = x + 1$ such that $f(y) = x$. This implies that g is onto.

Problem 2.2

a. $f \circ g(x) = f(g(x)) = f(x + 2) = (x + 2)^2 + 1 = x^2 + 4x + 5$

b. Yes, it has to be one-one.

   We will show that if g were not one-one, then $f \circ g$ cannot be one-one either.

   Let g be a function from set A to B, f a function from B to C, so that $f \circ g$ is a function from A to C.

   Suppose g were not one-one. Then, there are two elements $x, y \in A$ such that $g(x) = g(y)$. Then, $f \circ g(x) = f(g(x)) = f(g(y)) = f \circ g(y)$. Thus, $f \circ g$ is not one-one either.

Problem 2.3

We are given that n is an integer, and x a real number.

a. We will show this in two parts:

- Part 1: If $(x < n)$ then $\lfloor x \rfloor < n$.
  We know that $\lfloor x \rfloor \leq x$. Since $x < n$, it follows that $\lfloor x \rfloor < n$.
- If $\lfloor x \rfloor < n$ then $x < n$.
  We know that $\lfloor x \rfloor$ is an integer. If two integers differ, then their difference is at least 1. Since $\lfloor x \rfloor < n$, and both $\lfloor x \rfloor$ and n are integers, we have $\lfloor x \rfloor \leq n - 1$.
  We also know that $x < \lfloor x \rfloor + 1$ (we can verify this by cases, x is an integer and x is not an integer).
  Thus, we have $x < \lfloor x \rfloor + 1 \leq (n-1) + 1 = n$.

b. If x is any real number, show that $\lfloor -x \rfloor = -\lceil x \rceil$.

We consider two cases here:

- Case 1: x is an integer.
  In this case, $-x$ is an integer too.
  Thus, $\lfloor -x \rfloor = -x$, and $-\lceil x \rceil = -x$.
  The left hand side equals the right hand side.
- Case 2: x is not an integer.
  Suppose $a = \lfloor x \rfloor$ and $b = \lceil x \rceil$. We know that a and b are consecutive integers, and $a < x < b$.
  Thus, the right hand side is $-\lceil x \rceil = -b$.
  From $a < x < b$, we get $-a > -x > -b$ (note that for any two real numbers p, q, $p < q$ is equivalent to $-p > -q$). Since $-a$ and $-b$ are also consecutive integers, we know that $\lfloor -x \rfloor = -b$.
  Thus, the left hand side is $-b$, and we have proved that the right hand side equals the left hand side.

Problem 2.4

The left hand side is:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j &= \sum_{i=1}^{n}\{a_i b_1 + a_i b_2 + a_i b_3 + \ldots + a_i b_m\} \\ &= \sum_{i=1}^{n}\{a_i(b_1 + b_2 + b_3 + \ldots + b_m)\} \\ &= \sum_{i=1}^{n}\{a_i(S_b)\}\end{aligned}$$

where $S_b = b_1 + b_2 + \ldots + b_m = \sum_{j=1}^{m} b_i$.

Proceeding further, the left hand side is:

$$S_b \sum_{i=1}^{n} \{a_i\}$$

where we have removed the common factor S_b out, since it was a constant, independent of i. Note that if c is a constant, $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$.

Substituting S_b back into the expression, the left hand side reduces to:

$$\sum_{j=1}^{m} b_i \sum_{i=1}^{n} a_i$$

which equals the right hand side.

Problem 2.5

a. Denote the requires sum by S.

$$S = 1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \ldots + n \times 2^n$$

Multiply both sides by 2, we get:

$$2S = 1 \times 2^2 + 2 \times 2^3 + 3 \times 2^4 + \ldots + n \times 2^{n+1}$$

Taking the different of the two equations,

$$S - 2S = 1\times2 + (2\times2^2 - 1\times2^2) + (3\times2^3 - 2\times2^3) + \ldots + (n\times2^n - (n-1)\times2^n) - n\times2^{n+1}$$

$$\begin{aligned} -S &= (2 + 2^2 + 2^3 + \ldots + 2^n) - n \times 2^{n+1} \\ &= (2^{n+1} - 2 - n2^{n+1}) \\ &= -2 - (n-1)2^{n+1} \end{aligned}$$

where we used the identity $1 + 2 + 2^2 + \ldots + 2^n = 2^{(n+1)} - 1$.

Thus, we get $S = 2 + (n-1)2^{(n+1)}$.

b. The key to the solution is the identity

$$\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

Using this in the sum, we get

$$\begin{aligned}
\sum_{k=2}^{n} \frac{1}{k(k-1)} &= \sum_{k=2}^{n} \{ \frac{1}{k-1} - \frac{1}{k} \} \\
&= (1/1 - 1/2) + (1/2 - 1/3) + \ldots + (1/(n-1) - 1/n) \\
&= 1/1 + (-1/2 + 1/2) + (-1/3 + 1/3) + \ldots \\
&\quad +(-1/(n-1) + 1/(n-1)) - 1/n \\
&= 1 - 1/n
\end{aligned}$$

The final result is $1 - 1/n$.