Boundedness & Ultimate Boundedness

Def: $\dot{x} = f(t,x)$ has solution that is

- uniformly bounded if $\exists\, c>0$ s.t. $\forall\, a\in(0,c)$, $\exists\, \beta \equiv \beta(a)$: $\|x(t_0)\| \le a \Rightarrow \|x(t)\| \le \beta,\ \forall t \ge t_0$
- globally uniformly bdd if uniformly bdd with any c.
- uniformly ultimately bdd if $\exists\, b, c>0$ st. $\forall\, a\in(0,c)$, $\exists\, T \equiv T(a,b)$: $\|x(t_0)\| \le a \Rightarrow \|x(t)\| \le b,\ \forall t \ge t_0 + T$
- globally uniformly ultimately bdd if above holds for any c.

Use of Lyapunov fn. Suppose $V: D \Rightarrow \mathbb{R}$ cont. diff., +ve definite. Let $c>0$ s.t. Ω_c bdd.

Let $\Lambda = \{x\in \Omega_c \mid V(x) \ge \varepsilon\}$ and suppose $\dot{V}(t,x) \le -W_3(x),\ \forall x\in\Lambda,\ t \ge t_0$, where W_3 is cont. and +ve-definite.

While in Λ, trajectory behaves as if origin is uniformly asym. stable and satisfies,

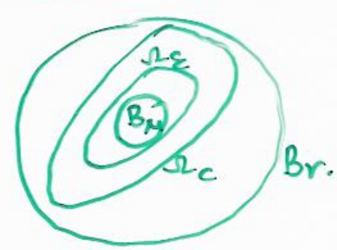
$$\|x(t)\| \le \beta(\|x(t_0)\|, (t-t_0)) \quad \text{for some } \mathcal{KL} \text{ fn. } \beta.$$

Thus $x(t)$ decreases continuously and eventually enters Ω_ε. To see this, let $k = \min_{x\in\Lambda} W_3(x)$. Due to cont. of W_3 and compactness of Λ, $k>0$ exists.

So, $\dot{V}(t,x) \le -W_3(x) \le -k \Rightarrow V(x(t)) \le V(x(t_0)) - k(t-t_0) \le c - k(t-t_0)$

$\Rightarrow V(x(t))$ reduces to ε within the interval $[t_0, t_0 + (c-\varepsilon)/k]$.

In some cases we have, $\dot{V}(t,x) \le -W_3(x)$ $\forall\, \mu \le \|x\| \le r$, $\forall\, t \ge t_0$. Then we can find Λ using the definitions of V.



$V>0 \Rightarrow \exists\, \alpha_1, \alpha_2 \in \mathcal{K}_r$: $\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$.

$x\in\Omega_c \Leftrightarrow V(t,x) \le c \Rightarrow \alpha_1(\|x\|) \le c \Leftrightarrow \|x\| \le \alpha_1^{-1}(c)$.

Choosing $c = \alpha_1(r)$ gives, $x\in\Omega_c \Rightarrow \|x\| \le r \Leftrightarrow x\in B_r$.

Also, $x\in B_\mu \Leftrightarrow \|x\| \le \mu \Rightarrow V(t,x) \le \alpha_2(\|x\|) \le \alpha_2(\mu)$. Choosing $\varepsilon = \alpha_2(\mu)$ gives, $x\in B_\mu \Rightarrow x\in\Omega_\varepsilon$. To have $\varepsilon < c$, we must have, $\mu < \alpha_2^{-1}(\alpha_1(r))$.

Thm: $V: [0,\infty) \times D \to \mathbb{R}$ cont. diff s.t. $\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$

$\dot{V} \le -W_3(x)$ $\forall\, \|x\| > \mu > 0$, $\forall\, t \ge 0$. W_3 is cont., +ve definite; $(\alpha_1, \alpha_2 \in \mathcal{K})$.

Suppose $r>0$: $B_r \subset D$ and choose $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then

$\|x(t_0)\| \le \alpha_2^{-1}(\alpha_1(r)) \Rightarrow \exists\, T$: $\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0)$ $\forall\, t_0 \le t \le t_0+T$

$\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu))$ $\forall\, t \ge t_0+T$

* $D = \mathbb{R}^n$, $\alpha_1 \in \mathcal{K}_\infty \Rightarrow$ above holds for any initial condition.

Input-to-State Stable

So far we considered autonomous systems. In presence of inputs (disturbance for example) would stability prevail? What additional conditions required assuming input is bounded?

Consider linear system: $\dot{x} = Ax + Bu \Rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

$$\Rightarrow \|x(t)\| \le k e^{-\lambda t} \|x(0)\| + \int_0^t k e^{-\lambda(t-\tau)} \|Bu(\tau)\| d\tau \qquad \underbrace{(\|e^{At}\| \le k e^{-\lambda t})}_{A \text{ Hurwitz}}$$

$$\le K e^{-\lambda t} \|x(0)\| + \frac{K\|B\|}{\lambda} \sup_{\tau \in [0,t]} \|u(\tau)\|$$

Thus exp. stable $\Rightarrow$ ISS (bounded input will keep state bounded).

The above property not enjoyed by a nonlinear system: $\dot{x} = -3x + (1+2x^2)u$

When $u=0$, $\dot{x} = -3x$ is exp. stable. Suppose $x(0) = 2$ and $u(t) \equiv 1$. Then

$x(t) = \dfrac{3-e^t}{3-2e^t}$ is unbounded and has a finite escape time.

Def: $\dot{x} = f(t, x, u)$ is ISS if $\exists\, \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$: $\forall t \ge t_0$, $\forall x(t_0)$, $\forall u(t)$:

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{\tau \in [t_0, t]} \|u(\tau)\|\right).$$

Locally ISS if above holds for all $\|x(t_0)\| < k_1$ and for all $\sup_{t \ge t_0} \|u(t)\| < k_2$.

Thm: $V: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ cont. diff. s.t. (i) $\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$ $\quad (\alpha_1, \alpha_2 \in \mathcal{K}_\infty)$,

(ii) $\dot{V}(t,x) \le -W_3(x)$, $\forall \underline{\|x\| \ge \rho(\|u\|)}$ $\quad$ (W_3 cont., +ve definite; $\rho \in \mathcal{K}$).

$\Rightarrow$ system is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Locally ISS if (i) & (ii) hold for $\|x\| < r$ and $\|u\| < r'$ and $\alpha_1, \alpha_2 \in \mathcal{K}$.

Example: $\dot{x} = -x^3 + u$. $u = 0 \Rightarrow \dot{x} = -x^3$ is globally asym. stable.

Let $V(x) = \frac{x^2}{2} \Rightarrow V > 0$ and radially unbdd. Also, $\dot{V} = x[-x^3 + u] = -x^4 + ux$

$$= -(1-\theta)x^4 - \theta x^4 + xu \quad (0 < \theta < 1).$$

$$\le -(1-\theta)x^4, \quad \forall |x| \ge \left(\frac{|u|}{\theta}\right)^{1/3}$$

So ISS with $\gamma(r) = \left(\frac{r}{\theta}\right)^{1/3}$.

Thm: $f(t,x,u)$ cont. diff. & globally Lipschitz in (x,u).

$\dot{x} = f(t,x,0)$ globally exp. stable $\Rightarrow$ $\dot{x} = f(t,x,u)$ ISS.

Input-to-Output stability

- Does bdd input (disturbance) produce bdd output? ISS guarantees bddness of state, but what about bddness of output?

- $\mathcal{L}^m = \{u : [0,\infty) \to \mathbb{R}^m \mid \|u\|_{\mathcal{L}} < \infty\}$; $\mathcal{L}^m_e = \{u \mid u_\tau \in \mathcal{L}^m, \forall \tau \geq 0\}$.
  ↑ truncation of u.

- **Def**: $H : \mathcal{L}^m_e \to \mathcal{L}^q_e$ is $\mathcal{L}$-stable if $\exists \alpha \in \mathcal{K}, \beta \geq 0 : \|(Hu)_\tau\|_{\mathcal{L}} \leq \alpha(\|u_\tau\|_{\mathcal{L}}) + \beta$ $\forall u \in \mathcal{L}^m_e, \tau \geq 0$

  is finite-gain $\mathcal{L}$-stable if $\exists \gamma, \beta \geq 0 : \|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma \|u_\tau\|_{\mathcal{L}} + \beta$, $\forall u \in \mathcal{L}^m_e, \tau \geq 0$.

  Small-signal $\mathcal{L}$-stable (resp. small-signal finite-gain $\mathcal{L}$-stable) if above holds $\forall u \in \mathcal{L}^m_e$ s.t. $\sup \|u(t)\| < r$.

Thm: $\dot{x} = f(t,x,0)$ is exp. stable and $\exists L, \eta_1, \eta_2 \geq 0$: $\begin{cases} \text{(i) } \|f(t,x,u) - f(t,x,0)\| \leq L\|u\| \\ \text{(ii) } \|h(t,x,u)\| \leq \eta_1 \|x\| + \eta_2 \|u\| \end{cases}$ $\forall t \in [0,\infty), \forall x \in D, \forall u \in D_u$

$\Rightarrow \dot{x} = f(t,x,u)$ is small-signal $\mathcal{L}_p$ finite-gain stable ($\forall p \in [1,\infty]$)

Additionally if $\dot{x} = f(t,x,0)$ globally asy. stable and $D = \mathbb{R}^n$, $D_u = \mathbb{R}^m \Rightarrow \dot{x} = f(t,x,u)$ finite-gain $\mathcal{L}_p$ stable ($\forall p \in [1,\infty]$).

Corollary: Linear system (A,B,C,D) is finite-gain stable if A is Hurwitz.

Example: $\begin{cases} \dot{x} = -x - x^3 + u \\ y = \tanh(x) + u \end{cases}$ For $\dot{x} = -x - x^3$ we can use $V(x) = \frac{x^2}{2}$ to show global exp. stable.

Also, $\left.\begin{aligned} &\|f(x,u) - f(x,0)\| = \|u\| \\ &\|h(x,u)\| \leq \|\tanh(x)\| + \|u\| \leq \|x\| + \|u\| \end{aligned}\right\} \Rightarrow L = \eta_1 = \eta_2 = 1.$

So, finite-gain stable.

Thm: $\dot{x} = f(t,x,0)$ is uniformly asy. stable and $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}, \eta \geq 0$: $\begin{cases} \text{(i) } \|f(t,x,u) - f(t,x,0)\| \leq \alpha_1(\|u\|) \\ \text{(ii) } \|h(t,x,u)\| \leq \alpha_2(\|x\|) + \alpha_3(\|u\|) + \eta \end{cases}$ for $t \in [0,\infty)$, $x \in D$, $u \in D_u$

(weaker assump.; weaker conclusion).

$\Rightarrow \dot{x} = f(t,x,u)$ small-signal $\mathcal{L}_\infty$-stable

Additionally $\mathcal{L}_\infty$-stability if $\dot{x} = f(t,x,u)$ is ISS, (ii), $D = \mathbb{R}^n$, $D_u = \mathbb{R}^m$.

Input-Output Stability

Corollary (to previous thm and earlier stability results):

$f(t,x,u)$ cont. diff. in nbhd. of $(x=0, u=0)$, $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial u}$ bdd uniformly in t,
$\exists\, \alpha_1, \alpha_2 \in \mathcal{K}, \eta > 0 : \|h(t,x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta$
$\dot{x} = f(t,x,u)$ unif. asym. stable $\Rightarrow$ $\dot{x} = f(t,x,u)$ small-signal $\mathcal{L}_\infty$-stable.

To have $\mathcal{L}_\infty$-stability (i.e., input need not be small-signal), global uniform asym. stability in previous thm is not enough. Need (global) ISS.

Thm: $\dot{x} = f(t,u,x)$ is ISS for all $x(t_0) \in \mathbb{R}^n$ and $u[t_0,\infty) \subseteq \mathbb{R}^m$,
$\exists\, \alpha_1, \alpha_2 \in \mathcal{K}, \eta > 0 : \|h(t,x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta$
$\Rightarrow$ $\dot{x} = f(t,x,u)$ is $\mathcal{L}_\infty$-stable.

Example: $\left.\begin{aligned} \dot{x} &= -x - 2x^3 + (1+x^2)u^2 \\ y &= x^2 + u \end{aligned}\right\}$ $u = 0 \Rightarrow \dot{x} = -x - 2x^3$ is globally exp. stable.

Let $V = \frac{x^2}{2}$ ($\Rightarrow V > 0$) and $\left.\begin{aligned} \dot{V} &= x[-x - 2x^3 + (1+x^2)u^2] = -x^2 - 2x^4 + xu^2 + x^3u^2 \\ &= -x(x-u^2) - x^3(x-u^2) - x^4 \le -x^4, \quad \forall |x| \ge u^2. \end{aligned}\right] \Rightarrow$ ISS

$\|x^2 + u\| \le |x|^2 + |u| \le \alpha_1(\|x\|) + \alpha_2(\|u\|)$. $\Rightarrow$ $\mathcal{L}_\infty$-stable.

$\mathcal{L}_2$ - Stability / Gain

$\mathcal{L}_\infty$-stability implies bdd output for bdd input. $\mathcal{L}_2$-stability requires finite-energy output for finite-energy input. Results on finite-gain $\mathcal{L}_p$-stability ($p \in [1,\infty]$) can be used. But question we want to answer is what is the finite gain when $p=2$. This is useful in optimal control design that minimizes $\mathcal{L}_2$-gain.

Linear Time-Inv. System: $\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\}$ with A Hurwitz ($\Rightarrow$ finite gain $\mathcal{L}_2$-stable).

$\mathcal{L}_2$-gain is given by, $\sup_\omega \|G(j\omega)\|_2$, where $G(s) = C(sI-A)^{-1}B + D$.

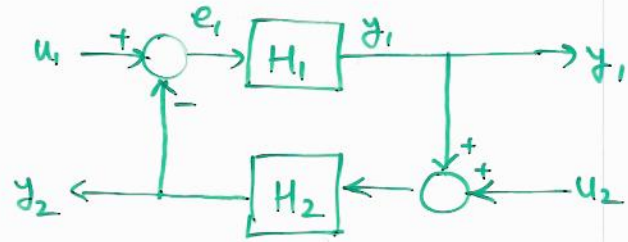
Proof: $\|y\|_2^2 = \int_0^\infty y^T(t)\, y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)\, Y(j\omega)\, d\omega$ (Parseval's Thm)

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)\, G^T(-j\omega)\, G(j\omega)\, U(j\omega)\, d\omega$ $\quad (Y(j\omega) = G(j\omega)\, U(j\omega))$.

$\le \left(\sup_\omega \|G(j\omega)\|\right)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)\, U(j\omega)\, d\omega = \left(\sup_\omega \|G(j\omega)\|\right)^2 \|u\|_2^2$.

Thus $\sup_\omega \|G(j\omega)\|$ is upper bound on $\mathcal{L}_2$-gain. Exactness of upper bound shown by contradiction
specific input

Small-Gain Thm for feedback Systems



Consider interconnection of two systems, where both are finite-gain $\mathcal{L}$-stable.
Supposing the interconnection is well-defined, i.e, $\forall\, u_1, u_2$, $\exists$ unique e_1, y_1, e_2, y_2.
Then under what condition the feedback system is finite-gain $\mathcal{L}$-stable?

Thm (small-gain thm): $\|y_{i\tau}\|_{\mathcal{L}} \le \gamma_i \|e_{i\tau}\|_{\mathcal{L}} + \beta_i \quad , \forall e_i \in \mathcal{L}_e^{m_i}, \tau \in [0,\infty)$.
Then feedback system is finite-gain $\mathcal{L}$-stable if $\gamma_1 \gamma_2 < 1$ (loop-gain <1).

Proof: $e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau \quad \& \quad e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau$.

So,
$$\begin{aligned}\|e_{1\tau}\| &\le \|u_{1\tau}\| + \|(H_2 e_2)_\tau\| \le \|u_{1\tau}\| + \gamma_2 \|e_{2\tau}\| + \beta_2 \\ &\le \|u_{1\tau}\| + \gamma_2 \left[\|u_{2\tau}\| + \gamma_1 \|e_{1\tau}\| + \beta_1 \right] + \beta_2 \\ &= \gamma_1 \gamma_2 \|e_{1\tau}\| + \left[\|u_{1\tau}\| + \gamma_2 \|u_{2\tau}\| + \gamma_2 \beta_1 + \beta_2 \right].\end{aligned}$$

Since $1 - \gamma_1\gamma_2 > 0$, $\|e_{1\tau}\| \le \dfrac{1}{1-\gamma_1\gamma_2} \left[\|u_{1\tau}\| + \gamma_2 \|u_{2\tau}\| + \gamma_2\beta_1 + \beta_2 \right]$. $\forall \tau \in [0,\infty)$.

Similarly, $\|e_{2\tau}\| \le \dfrac{1}{1-\gamma_1\gamma_2} \left[\|u_{2\tau}\| + \gamma_1 \|u_{1\tau}\| + \gamma_1 \beta_2 + \beta_1 \right]$.

Finally, $\|e_\tau\| \le \|e_{1\tau}\| + \|e_{2\tau}\|$.