Exp. stable if $\exists c, k, \lambda > 0$: $\|x(t)\| \le k \|x(t_0)\| e^{-\lambda(t-t_0)}$ $\forall \|x(t_0)\| \le c$.
Globally exp. stable if exp. stable for any initial condition.

Comparison Functions

Class $\mathcal{K}_a$: $\alpha : [0, a) \to [0, \infty)$ s.t. $\alpha(0) = 0$, α strictly increasing
Class $\mathcal{K}_\infty$: $\alpha : [0, \infty) \to [0, \infty)$ s.t. $\alpha(0) = 0$, α strictly increasing, $\alpha(r) \xrightarrow[r \to \infty]{} \infty$
class $\mathcal{K}_a\mathcal{L}$: $\beta : [0, a) \times [0, \infty) \to [0, \infty]$ s.t. $\forall s \in [0, \infty)$: $\beta(r, s) \in \mathcal{K}_a$
$\forall r \in [0, a)$: $\beta(r, s)$ decreasing, $\xrightarrow[s \to \infty]{} 0$

$\alpha_1, \alpha_2 \in \mathcal{K}_a \Rightarrow \alpha_1 \circ \alpha_2 \in \mathcal{K}_a$; $\alpha_1, \alpha_2 \in \mathcal{K}_\infty \Rightarrow \alpha_1 \circ \alpha_2 \in \mathcal{K}_\infty$.
$\alpha \in \mathcal{K}_a \Rightarrow \alpha^{-1} \in \mathcal{K}_{\alpha(a)}$; $\alpha \in \mathcal{K}_\infty \Rightarrow \alpha^{-1} \in \mathcal{K}_\infty$.
$\beta \in \mathcal{K}_a\mathcal{L}$, $\alpha_1, \alpha_2 \in \mathcal{K}_a \Rightarrow \alpha_1(\beta(\alpha_2(r), s)) \in \mathcal{K}_a\mathcal{L}$.

$V : D \to \mathbb{R}$ cont, +ve-definite, $\exists r > 0$: $B_r \subset D$, $\Rightarrow \exists \alpha_1, \alpha_2 \in \mathcal{K}_r : \alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$
Special case: $\lambda_{min}(P)\|x\|^2 \le x^T P x \le \lambda_{max}(P)\|x\|^2$ when $P > 0$.
To have $B_\delta \subseteq \Omega_\beta \subseteq B_r$ $\boxed{\text{choose } \delta, \beta \text{ s.t.}}$ $\alpha_2(\delta) \le \beta \le \alpha_1(r)$. Then,
$[V(x) \le \beta \Rightarrow \alpha_1(\|x\|) \le \alpha_1(r) \Leftrightarrow \|x\| \le r] \Leftrightarrow [\Omega_\beta \subseteq B_r]$.
$[\|x\| \le \delta \Leftrightarrow \alpha_2(\|x\|) \le \alpha_2(\delta) \Rightarrow V(x) \le \beta] \Leftrightarrow [B_\delta \subseteq \Omega_\beta]$.

Also if $\dot{V} < 0 \Leftrightarrow -\dot{V} > 0$. Then $r > 0$ can be chosen s.t. $\exists \alpha_3, \alpha_4 \in \mathcal{K}_r : \alpha_3(\|x\|) \le -\dot{V} \le \alpha_4(\|x\|)$
$\Rightarrow +\dot{V} \le -\alpha_3(\|x\|)$. Also, $V \le \alpha_2(\|x\|) \Leftrightarrow \alpha_2^{-1}(V) \le \|x\| \Leftrightarrow \alpha_3(\alpha_2^{-1}(V)) \le \alpha_3(\|x\|)$.
$\Rightarrow \dot{V} \le -\alpha_3(\alpha_2^{-1}(V))$. $\Rightarrow V(x(t)) \le \beta(V(x(0)), t)$, where $\beta \in \mathcal{K}_r\mathcal{L}$.
$\Rightarrow V \xrightarrow[t \to \infty]{} 0$

Uses the result that, $\dot{y} = -\alpha(y) \Rightarrow y(t) = \beta(y_0, t - t_0)$. ($\alpha \in \mathcal{K}_r$, $\beta \in \mathcal{K}_r\mathcal{L}$).

Further, $V(x(t)) \le V(x(0)) \Rightarrow \alpha_1(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \alpha_2(\|x(0)\|)$
$\Rightarrow \|x(t)\| \le \alpha_1^{-1}(\alpha_2(\|x(0)\|))$.
Also, $V(x(t)) \le \beta(V(x(0)), t) \Rightarrow \alpha_1(\|x(t)\|) \le V(x(t)) \le \beta(V(x(0)), t) \le \beta(\alpha_2(\|x(0)\|), t)$
$\Rightarrow \|x(t)\| \le \alpha_1^{-1}[\beta(\alpha_2(\|x(0)\|), t)]$.

Thus using comparison functions facts used in Thm 4.1 can be derived, and even more can be deduced.

Stability for $\dot{x} = f(t, x)$ with $0 = f(t, 0)$, $\forall t \ge t_0$ (origin is eq. at $t = t_0$)
0 stable if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, t_0) > 0$: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$, $\forall t \ge t_0$ (sec top)
0 uniformly stable if $\delta = \delta(\varepsilon)$.
Similarly asy. stable, uniformly asymp. stable; globally uniformly asymp. stable.

Stability results for time-varying systems

Representation using comparison fns.

- uniformly stable $\Leftrightarrow \exists \alpha \in \mathcal{K}_c : \|x(t)\| \le \alpha(\|x(t_0)\|)$, $\forall t \geq t_0$, $\forall \|x(t_0)\| < c$.
- unif. asy. stable $\Leftrightarrow \exists \beta \in \mathcal{KL}: \|x(t)\| \le \beta(\|x(t_0)\|, t-t_0)$, $\forall t \ge t_0$, $\forall \|x(t_0)\| < c$.

Condition for uniform stability: $V: [0,\infty) \times D \to \mathbb{R}$ cont. diff. s.t. $\forall t \geq 0$, $\forall x \in D$:
i) $W_1(x) \le V(t,x) \le W_2(x)$, (ii) $\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f \le 0$, where $W_1, W_2 > 0$, conts on D.
Then unif. stable.

Condition for unif. asy. stability: $V: [0,\infty) \times D \to \mathbb{R}$, $W_1, W_2, W_3 > 0$ & cont. over D st. $\forall t \geq 0$, $\forall x \in D$: (i) $W_1(x) \le V(t,x) \le W_2(x)$, (ii) $\dot{V} \le -W_3(x)$. Then unif. asymp. stable.

Further for $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, $x(t_0) \in \Omega_c \Rightarrow \|x(t)\| \le \beta(\|x(t_0)\|, t-t_0)$, $\forall t \geq t_0 \geq 0$, $\beta \in \mathcal{KL}$.

Finally if $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded $\Rightarrow$ globally uniformly asymp. stable.

Condition for exp. stability: $V: [0,\infty) \times D \to \mathbb{R}$, $k_1, k_2, k_3, a > 0$ s.t.
$\forall t \geq 0$, $\forall x \in D$: (i) $k_1 \|x\|^a \le V(t,x) \le k_2 \|x\|^a$, (ii) $\dot{V} \le -k_3 \|x\|^a$
Then exp. stable. Further if $D = \mathbb{R}^n$, then globally exp. stable

Specialization to linear system / comparison to linear system or linearized system

(i) Suppose $f: [0,\infty) \times D$, where $D = \{x \mid \|x\| < r\}$, cont. diff. with $\frac{\partial f}{\partial x}$ bounded and Lipschitz on D. Then 0 is exp. stable for f iff 0 is exp. stable for $A(t) := \left.\frac{\partial f}{\partial x}\right|_{x=0}$.

(ii) For linear system, 0 is (globally) exp. stable iff 0 is (globally) unif. asy. stable iff $\exists k, \lambda > 0$: $\|\underbrace{\Phi(t,t_0)}_{\text{state-transition matrix}}\| \le k e^{-\lambda(t-t_0)}$ $\forall t \ge t_0 \geq 0$.

Note for nonlinear systems, exp. stability is stronger than unif. asymp. stability.

Converse Lyapunov Theorems: These establish the existence of a suitable Lyapunov fn. of the type discussed above under various stability properties and certain assumptions on the system.

(i) $f: [0,\infty) \times D$, where $D = \{x \mid \|x\| < r\}$, cont.-diff. with $\frac{\partial f}{\partial x}$ uniformly bdd on D. If 0 is unif. asym. stable, then $\exists V: [0,\infty) \times D_0 \to \mathbb{R}$ ($D_0 \subset D$ and depends on domain of unif. asy. stability) s.t. properties of V as in suff. condition above hold. (Similarly for exp. stability)

Examples of stability properties

Example 1:

Consider $\left.\begin{aligned}\dot{x}_1 &= -x_1 - g(t)\,x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}\right\}$ with $0 \le g(t) \le k$ and $\dot{g}(t) \le g(t)$, $\forall t \ge 0$

Let $V(t,x) = x_1^2 + [1+g(t)]\,x_2^2$. Then, $x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)\,x_2^2$.

So, $V(t,x) \ge x_1^2 + x_2^2 > 0$ (is "+ve definite" and is "radially unbounded").

$V(t,x) \le x_1^2 + (1+k)\,x_2^2$ (is "decrescent")

Also $\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f = \dot{g}(t)\,x_2^2 + 2x_1\,[-x_1 - \cancel{g(t)}\,x_2] + 2[1+g(t)]\,x_2(x_1 - x_2)$

$$= -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2g(t)\,x_2^2 - \dot{g}(t)\,x_2^2 = -2[x_1^2 - x_1x_2 + x_2^2(1+g(t)-\dot{g}(t))]$$

Since $1 + g(t) - \dot{g}(t) \ge 1$, $\dot{V}(t,x) \le -2[x_1^2 - x_1x_2 + x_2^2] = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Since $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0$, $\dot{V}(t,x) \le -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x < 0$.

It follows that the system is globally asym. stable.

Further since $\lambda_{min}(P)\|x\|^2 \le x^T P x \le \lambda_{max}(P)\|x\|^2$ for any $P > 0$, system is globally exp. stable.

Example 2:

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix} \Rightarrow \lambda(A(t)) = -.25 \pm .25\sqrt{7}\,j$$

(independent of t).

So eigen values of $A(t)$ in LHP for all t. However,

$$\Phi(t,0) = \begin{bmatrix} e^{.5t}\cos t & e^{-t}\sin t \\ -e^{.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$ which is unbounded $\Rightarrow$ not exp. stable.

Boundedness & Ultimate Boundedness

In some cases solution may remain bounded eventhough no eq. exists.

Example: $\dot{x} = -x + \delta\sin t$, $x(t_0) = a > \delta > 0$. Has no eq. pt.

Solution $x(t) = e^{-(t-t_0)}a + \delta\int_{t_0}^{t} e^{-(t-\tau)}\sin\tau\, d\tau$

$$\Rightarrow |x(t)| \le e^{-(t-t_0)}a + \delta\int_{t_0}^{t} e^{-(t-\tau)}\,d\tau = e^{-(t-t_0)}a + \delta[1 - e^{-(t-t_0)}]$$

$$= e^{-(t-t_0)}(a-\delta) + \delta \le a \quad \forall\, t \ge t_0.$$

Thus $x(t)$ is bounded; bound is uniform (does not depend on t_0).

The bound is conservative since does not consider the exp. decay. In fact,

$\forall\, b \in (\delta, a)$: $|x(t)| \le b$ $\forall\, t \ge t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$. b an "ultimate bound"