Centralized Control

Thm: Given plant G, desired behavior K, observation mask M.

1. There exists Σ_u-enabling, non-marking, M-compatible supervisor S' s.t. $L(G||S) = K$ iff $K = pr(K) \neq \emptyset$, K controllable and observable.

2. There exists Σ_u-enabling, non-marking, non-blocking, M-compatible S s.t. $L_m(G||S) = K$ iff $K = pr(K) \cap L_m(G) \neq \emptyset$, K controllable and observable.

Example:

G

$\Sigma_u = \{b\}$
$M(a) = \varepsilon, \quad M(b) = b.$

- $K = pr(a^* b)$
- $s \in a^* \Rightarrow sb \in K$; $s \notin a^* \Rightarrow sb \notin L(G)$. So K controllable.
- Consider $s, t \in K$ s.t. $M(s) = M(t)$. Then
  either $s, t \in a^*$, and both a, b are enabled after s and t ;
  or $s, t \notin a^*$, and a is disabled after s and t.
  So K observable, and can be achieved as closed-loop behavior
- Consider another mask M' s.t. $M'(a) = M'(b) \neq \varepsilon$.
  Then $aa \in K$, $ba \in L(G)$ but $ba \notin K \Rightarrow K$ not observable.
  So M'-compatible supervisor for K does not exist.

$K = pr(a^* b)$
$\Rightarrow a, b \in pr(K)$
$M'(a) = M'(b)$
$aa \in pr(K)$
$ba \in L(G) - pr(K)$
$\Rightarrow K$ is not M'-observable

Test for observability

$s,t \in pr(K)$, $M(s) = M(t)$, $s\sigma \in pr(K)$, $t\sigma \in L(G)$ $\Rightarrow$ $t\sigma \in pr(K)$

$G = (X, \Sigma, \alpha, x_0, X_m)$ plant automaton

$S = (Y, \Sigma, \beta, y_0, Y_m)$ trim acceptor for $K \subseteq L_m(G) \Rightarrow \begin{cases} L_m(S) = K \\ L(S) = pr(K) \end{cases}$

Construct a new state machine with states $(Y \times Y \times X) \cup \{NO\}$

events $\left[(\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\})\right] - \{(\varepsilon,\varepsilon)\}$

initial state (y_0, y_0, x_0)

marked states $Y_m \times Y_m \times X_m$

transitions:

$$(y_0, y_0, x_0) \xrightarrow{(s,t)} (y_1, y_2, x) \xrightarrow{(\sigma,\sigma)} NO$$

$$(y_1, y_2, x) \xrightarrow{(\sigma,\sigma')} (y_1', y_2', x') \quad \text{iff} \quad \begin{cases} \beta(y_1,\sigma) = y_1' \\ \beta(y_2,\sigma') = y_2' \\ \alpha(x,\sigma') = x' \\ M(\sigma) = M(\sigma') \end{cases}$$

$$(y_1, y_2, x) \xrightarrow{(\sigma,\sigma)} NO \quad \text{iff} \quad \begin{cases} \beta(y_1,\sigma),\ \alpha(x,\sigma) \text{ defined} \\ \beta(y_2,\sigma) \text{ not defined} \\ \sigma \neq \varepsilon \end{cases}$$

- 1st coordinate tracks s as automaton S
  2nd — " — t — " — S
  3rd — " — t — " — G

- K not observable iff "NO" is reachable in this automaton

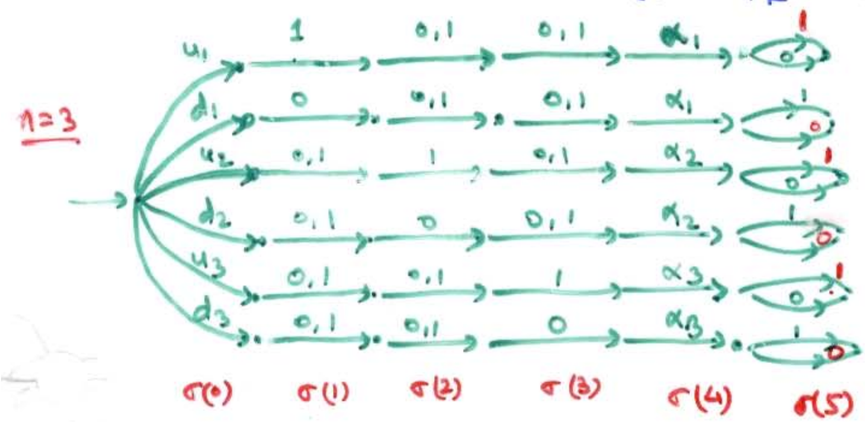
- complexity $O(|G|\,|S|^2) = O(mn^2)$

HW: Consider two examples of previous page. Apply test for observability as above.

Unless $P = NP$, off-line supervisor computation is not of polynomial complexity (in case of partial observation) (Tsitsiklis '89)

Consider a plant G with $\Sigma = \{u_1, \dots u_n\} \cup \{d_1, \dots, d_n\} \cup \{0,1\} \cup \{\alpha_1, \dots, \alpha_n\}$

$$L(G) = pr\left\{ \sigma(0) \dots \sigma(n+2) \;\middle|\; \begin{array}{l} \sigma(0) \in \{u_1, \dots, u_n\} \cup \{d_1, \dots, d_n\} \\ \sigma(i) \in \{0,1\} \quad i = 1, \dots, n, \; n+2 \\ \sigma(n+1) \in \{\alpha_1, \dots \alpha_n\} \\ \sigma(0) = u_K \Rightarrow \sigma(K) = 1, \; \sigma(n+1) = \alpha_K \\ \sigma(0) = d_K \Rightarrow \sigma(K) = 0, \; \sigma(n+1) = \alpha_K \end{array} \right\}$$



# of states in G = # of states in K

$= 1 + 2n(n+3) = O(n^2)$

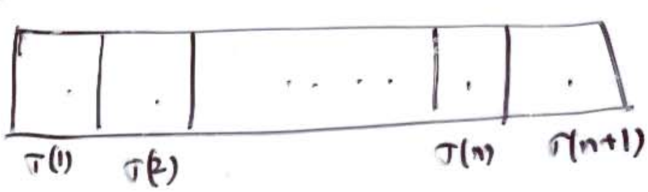
Desired behavior $K = pr\left\{ \sigma(0) \dots \sigma(n+2) \in L(G) \;\middle|\; \begin{array}{l} \sigma(0) = u_K \Rightarrow \sigma(n+2) = 1 \\ \sigma(0) = d_K \Rightarrow \sigma(n+2) = 0 \end{array} \right\}$

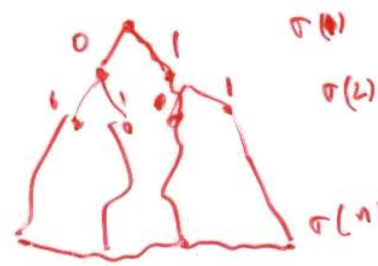
$M(u_K) = M(d_K) = \varepsilon$; $M(\sigma) = \sigma \quad \sigma \in \{0,1\} \cup \{\alpha_1, \dots \alpha_n\}$.

M-compatible supervisor remembers $s = \sigma(1) \dots \sigma(n+1)$

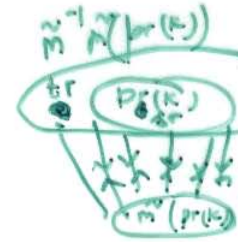
$$\text{if } \sigma(n+1) = \alpha_K \Rightarrow \begin{cases} \sigma_K = 1 & \Rightarrow \text{enable } 1 \text{ (disable } 0) \text{ after } s \\ \sigma_K = 0 & \Rightarrow \text{enable } 0 \text{ (disable } 1) \text{ after } s \end{cases}$$

The supervisor has $O(2^n)$ states since at time $(n+1)$ it remembers n bits of information.





Alternate Characterization of Observability



- Define $\tilde{M} : \Sigma \to \Delta\Sigma \cup \{\epsilon\}$

  $\tilde{M}(\epsilon) := \epsilon$, $\quad \tilde{M}(s\sigma) := M(s)\sigma$

  $\tilde{M}$ masks all but the last event

- $\tilde{M}^{-1}\tilde{M}(\epsilon) = \epsilon$; $\tilde{M}^{-1}\tilde{M}(s\sigma) = \{t\sigma \mid M(t) = M(s)\}$.

- Thm: K observable iff $\boxed{\overbrace{\sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right] \cap L(G)}^{H} \subseteq pr(K)}$.

Proof: $K = \phi$, then assertion trivially holds. So assume $K \neq \phi$.

($\Rightarrow$) Suppose for contradiction $H \not\subseteq pr(K)$. Pick smallest $s \in H - pr(K)$. $s \neq \epsilon$.

Let $s = \bar{s}\sigma$. $\bar{s}\sigma \in H \Rightarrow \bar{s}\sigma \in \tilde{M}^{-1}M(pr(K)) \supseteq H$

$\Rightarrow$ exists $\bar{t}\sigma \in pr(K)$ s.t. $M(\bar{t}) = M(\bar{s})$, contradicts observability of K

($\Leftarrow$) Pick $s\sigma \in pr(K)$, $t\sigma \in L(G)$, with $t \in pr(K)$, $M(s) = M(t)$. Need to show $t\sigma \in pr(K)$.

Since $H \subseteq pr(K)$ and $t\sigma \in L(G)$, suffices to show $t\sigma \in \sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right]$.

i.e., each prefix of $t\sigma \in \tilde{M}^{-1}\tilde{M}(pr(K))$.

Since $t \in pr(K)$, each prefix of $t \in pr(K) \supseteq \tilde{M}^{-1}\tilde{M}(pr(K))$.

Suffices to show $t\sigma \in \tilde{M}^{-1}\tilde{M}(pr(K))$

This follows from $s\sigma \in pr(K)$ and $M(s) = M(t)$.

- Remember:

K observable iff $\sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right] \cap L(G) \subseteq pr(K)$

K normal iff $M^{-1}M(pr(K)) \cap L(G) \subseteq pr(K)$