

Section 8.6

Partial Orderings

Definition: Let R be a relation on A . Then R is a *partial order* iff R is

- reflexive
 - antisymmetric
- and
- transitive

(A, R) is called a partially ordered set or a *poset*.

Note: It is not required that two things be related under a partial order. That's the *partial* part of it.

If two objects are always related in a poset, it is called a *total order* or *linear order* or *simple order*. In this case (A, R) is called a *chain*.

Examples:

- (\mathbb{Z}, \leq) is a poset. In this case either $a \leq b$ or $b \leq a$ so two things are always related. Hence, (\mathbb{Z}, \leq) is a total order and (\mathbb{Z}, \leq) is a chain.

- If S is a set then $(P(S), \subseteq)$ is a poset. It may not be the case that $A \subseteq B$ or $B \subseteq A$. Hence, \subseteq is not a total order.

- $(\mathbb{Z}^+, \text{'divides'})$ is a poset which is not a chain.

Definition: Let R be a total order on A and suppose $S \subseteq A$. An element s in S is a *least element* of S iff sRb for every b in S .

Similarly for *greatest* element.

Note: this implies that $\langle a, s \rangle$ is not in R for any a unless $a = s$. (There is nothing smaller than s under the order R).

Definition: A chain (A, R) is *well-ordered* iff every subset of A has a least element.

Examples:

- (\mathbb{Z}, \leq) is a chain but not well-ordered. \mathbb{Z} does not have least element.
 - (\mathbb{N}, \leq) is well-ordered.
 - (\mathbb{N}, \geq) is not well-ordered.
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Lexicographic Order

Given two posets (A_1, R_1) and (A_2, R_2) we construct an *induced* partial order R on $A_1 \times A_2$:

$\langle x_1, y_1 \rangle R \langle x_2, y_2 \rangle$ iff

- $x_1 R_1 x_2$

or

- $x_1 = x_2$ and $y_1 R_2 y_2$.

Example:

Let $A_1 = A_2 = \mathbb{Z}^+$ and $R_1 = R_2 = \text{'divides'}$.

Then

- $\langle 2, 4 \rangle R \langle 2, 8 \rangle$ since $x_1 = x_2$ and $y_1 R_2 y_2$.
- $\langle 2, 4 \rangle$ is not related under R to $\langle 2, 6 \rangle$ since $x_1 = x_2$ but 4 does not divide 6.
- $\langle 2, 4 \rangle R \langle 4, 5 \rangle$ since $x_1 R_1 x_2$. (Note that 4 is not related to 5).

This definition extends naturally to multiple Cartesian products of partially ordered sets:

$$A_1 \times A_2 \times A_3 \times \dots \times A_n.$$

Example: Using the same definitions of A_i and R_i as above,

- $\langle 2, 3, 4, 5 \rangle R \langle 2, 3, 8, 2 \rangle$ since $x_1 = x_2$, $y_1 = y_2$ and 4 divides 8.

- $\langle 2, 3, 4, 5 \rangle$ is not related to $\langle 3, 6, 8, 10 \rangle$ since 2 does not divide 3.

Strings

We apply this ordering to strings of symbols where there is an underlying 'alphabetical' or partial order (which is a total order in this case).

Example:

Let $A = \{ a, b, c \}$ and suppose R is the natural alphabetical order on A :

$$a R b \text{ and } b R c.$$

Then

- Any shorter string is related to any longer string (comes before it in the ordering).
- If two strings have the same length then use the induced partial order from the alphabetical order:

$aabc \ R \ abac$

Hasse or Poset Diagrams

To construct a Hasse diagram:

- 1) Construct a digraph representation of the poset (A, R) so that all arcs point up (except the loops).
- 2) Eliminate all loops
- 3) Eliminate all arcs that are redundant because of transitivity
- 4) eliminate the arrows at the ends of arcs since everything points up.

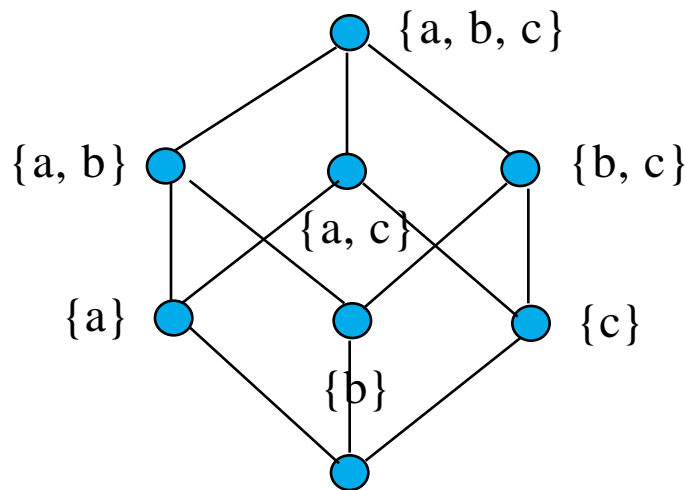
Example:

Construct the Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

The elements of $P(\{a, b, c\})$ are

$$\begin{aligned} &\{a\}, \{b\}, \{c\} \\ &\{a, b\}, \{a, c\}, \{b, c\} \\ &\{a, b, c\} \end{aligned}$$

The digraph is



Maximal and Minimal Elements

Definition: Let (A, R) be a poset. Then a in A is a *minimal element* if there does not exist an element b in A such that bRa .

Similarly for a *maximal element*.

Note: there can be more than one minimal and maximal element in a poset.

Example: In the above Hasse diagram, a is a minimal element and $\{b, c\}$ is a maximal element.

Least and Greatest Elements

Definition: Let (A, R) be a poset. Then a in A is the *least element* if for every element b in A , aRb and b is the *greatest element* if for every element a in A , aRb .

Theorem: Least and greatest elements are unique.

Proof:

Assume they are not. . .

Example:

In the poset above a is the greatest element. b is the least element.

Upper and Lower Bounds

Definition: Let S be a subset of A in the poset (A, R) . If there exists an element a in A such that sRa for all s in S , then a is called an *upper bound*. Similarly for lower bounds.

Note: to be an upper bound you must be related to every element in the set. Similarly for lower bounds.

Example:

- In the poset above, $\{a, b, c\}$, is an upper bound for all other subsets. $\{d, e, f\}$ is a lower bound for all other subsets.

Least Upper and Greatest Lower Bounds

Definition: If a is an upper bound for S which is related to all other upper bounds then it is the *least upper bound*, denoted $\text{lub}(S)$. Similarly for the *greatest lower bound*, $\text{glb}(S)$.

Example:

Consider the element $\{a\}$.

Since

$$\{a, b, c\}, \{a, b\}, \{a, c\} \text{ and } \{a\}$$

are upper bounds and $\{a\}$ is related to all of them, $\{a\}$ must be the lub. It is also the glb.

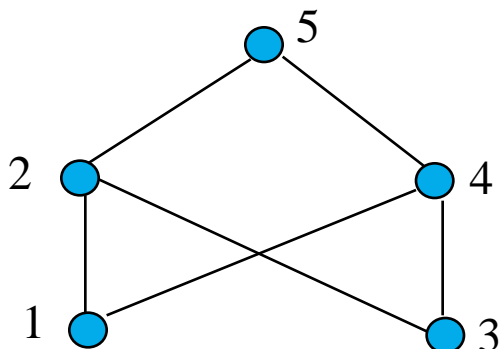
Lattices

Definition: A poset is a *lattice* if every pair of elements has a lub and a glb.

Examples:

- In the poset $(P(S), \subseteq)$, $\text{lub}(A, B) = A \cup B$. What is the $\text{glb}(A, B)$?

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Consider the elements 1 and 3.

- Upper bounds of 1 are 1, 2, 4 and 5.
- Upper bounds of 3 are 3, 2, 4 and 5.
- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
 - 2 is not related to 4
 - 4 is not related to 2
 - 2 and 4 are both related to 5.
- There is no glb either.

The poset is not a lattice.

Topological Sorting

We impose a total ordering R on a poset *compatible* with the partial order.

- Useful in PERT charts to determine an ordering of tasks
- Useful in rendering in graphics to render objects from back to front to obscure hidden surfaces

- A painter uses a topological sort when applying paint to a canvas - he/she paints parts of the scene furthest from the view first

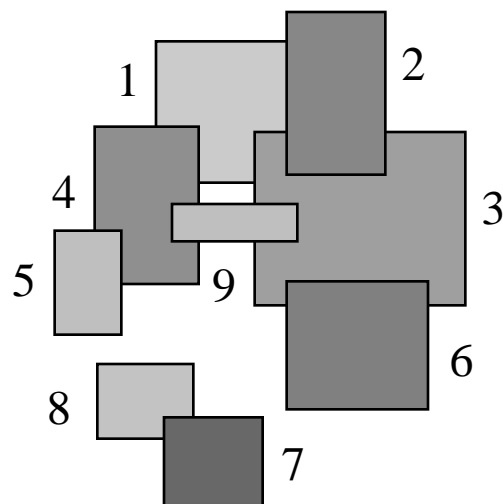
Algorithm: To sort a poset (S, R) .

- Select a (any) minimal element and put it in the list. Delete it from S .
- Continue until all elements appear in the list (and S is void).

Example:

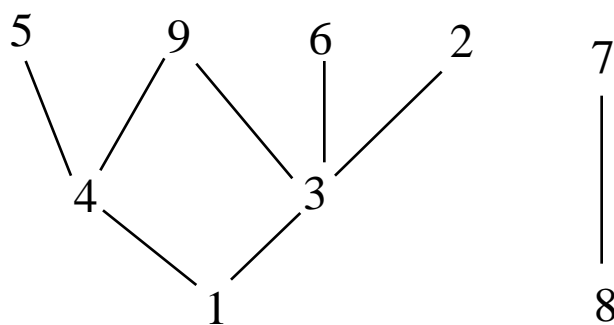
Consider the rectangles T and the relation $R =$ “is more distant than.” Then R is a partial order on the set of rectangles.

Two rectangles, T_i and T_j , are related, $T_i R T_j$, if T_i is more distant from the viewer than T_j .

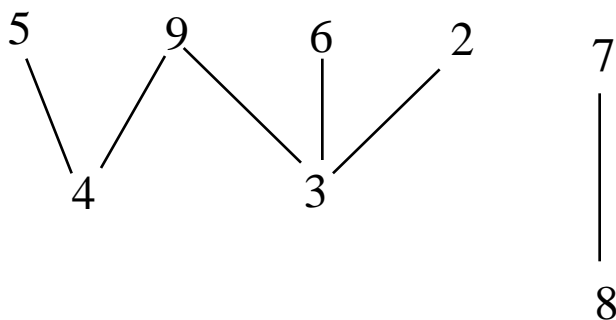


Then $1R2$, $1R4$, $1R3$, $4R9$, $4R5$, $3R2$, $3R9$, $3R6$, $8R7$.

The Hasse diagram for R is



Draw 1 (or 8) and delete 1 from the diagram to get



Now draw 4 (or 3 or 8) and delete from the diagram.
 Always choose a minimal element. Any one will do.

...and so forth.

