Section 7.4
Generating Functions

Generating functions are useful for manipulating sequences and therefore for solving counting problems.

Definition: Let S = $\{a_0, a_1, a_2, a_3, \ldots\}$ be an (infinite) sequence of real numbers. Then the *generating function* *G(x),* of S is the series

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

Note: For students of calculus -

> There is no issue of convergence here. The variable x and its powers allow us to assign a position to the numbers a_k. We will use the closed form expressions which represent G if the series converges but for symbol manipulation purposes only.

Examples:

- Let S = {1, 1, 1, 1, ... }. Then

$$G(x) = 1 + x + x^2 + x^3 + ... + x^k + ... = \sum_{k=0}^{\infty} x^k$$

• Let S = {0, 1, 2, 3, 4, 5, ...}. Then

$$G(x) = \sum_{k=0}^{\infty} kx^k$$

• If S is the finite sequence {1,1,1} then we pad S with an infinite number of zeros {1, 1, 1, 0, 0, ...} and we have

$$G(x) = 1 + x + x^2 = \frac{(1 - x^3)}{(1 - x)},$$

a closed form for G.

The Binomial Theorem Revisited

For producing expressions for G(x), recall the *binomial theorem*:

$$(a + b)^n = \frac{a^n}{0!} + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + ...$$

$$+ \frac{n(n-1)...(n-k+1)a^{n-k}b^k}{k!} + ... + \frac{n(n-1)...(n-n+`1)a^{n-n}b^n}{n!}$$

which terminates when n is an integer to produce a finite sum.

However, the formula can also be extended to include the case when n is not an integer:

- the sum does not terminate
- useful for producing expressions for generating functions.

Example:

We apply the procedure to the expression

$$\frac{1}{(1-3x^2)}$$

where we let $a=1, b=(-3x^2)$, and $n=-1$:

$$\frac{1}{(1-3x^2)} = (1-3x^2)^{-1} = \frac{1^{-1}}{0!} + \frac{(-1)1^{-2}(-3x^2)}{1!} +$$
$$\frac{(-1)(-2)1^{-3}(-3x^2)^2}{2!} + \frac{(-1)(-2)(-3)1^{-4}(-3x^2)^3}{3!} + \ldots + (3x^2)^k + \ldots$$

Since there are no odd powers of x, the expression is the generating function for the sequence

$$S = \{1,\ 0,\ 3,\ 0\ ,\ 3^2,\ 0\ ,\ 3^3,\ 0,\ \ldots\}$$

Manipulation of Generating Functions

Let

$$F(x)=\sum_{k=0}^{\infty}a_k x^k \text{ and } G(x)=\sum_{k=0}^{\infty}b_k x^k .$$

Then

- Sum

$$F(x)+G(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$$

- Product

$$F(x)G(x)=\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}a_j b_{k-j}\Big)x^k$$

To derive the expression we multiply each term of F by all terms of G and place the coefficients of like powers of x in the same column:

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots)$$

x^0	x^1	x^2	x^3 ...
$a_0 b_0$	$a_0 b_1$	$a_0 b_2$	$a_0 b_3$
	$a_1 b_0$	$a_1 b_1$	$a_1 b_2$
		$a_2 b_0$	$a_2 b_1$
			$a_3 b_0$
$\sum_{j=0}^{0} a_j b_{0-j}$	$\sum_{j=0}^{2} a_j b_{1-j}$	$\sum_{j=0}^{2} a_j b_{2-j}$	$\sum_{j=0}^{3} a_j b_{3-j}$ ...

Example:

A expression for the generating function of the sequence S = {1, 1, 1, 1, 1, 1, ...} is

$$\frac{1}{(1-x)}$$

Hence, the expression

$$\frac{1}{(1-x)^2}$$

is the generating function for the sequence

S = {1, 2, 3, 4, ... , k + 1, ...}.

Counting with Generating Functions

Recall that

$$x^a x^b = x^{a+b}$$

and therefore if

$$S = \{a_0, a_1, a_2, a_3, \ldots\} \text{ and } T = \{b_0, b_1, b_2, b_3, \ldots\}$$

there will be a contribution of $a_m b_n$ to the m + n power of x in the product of the generating functions for S and T.

Example:

Suppose we wish to count the integer solutions to a + b = 10 but a and b are constrained by

$$1 \le a \le 6 \text{ and } 3 \le b \le 9$$

There are 6 possible solutions:

1 + 9
2 + 8
3 + 7
4 + 6
5 + 5
6 + 4

where the first number is a and the second is b.

If we construct the sequences

A = {0, 1, 1, 1, 1, 1, 1, 0, 0, ...}

where a_k = 1 if k is a possible value of a and

B = {0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, ...}

where b_k = 1 if k is a possible value of b then the product of the two generating functions for A and B will have the total possible solutions to

a + b = 10

as a coefficient of x^{10}

$$\sum_{k=0}^{10} a_k b_{10-k}$$

= 0 + 1 + 1 + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 6

Solving Recurrence Relations

First some observations:

If

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

then

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$$

and

$$x^2 G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2} x^k$$

etc.

Example:

Solve the nonhomogeneous recurrence system

$$a_n = 3a_{n-2} + 1, a_0 = a_1 = 1.$$

Solution:

Multiply each term on both sides of the equation by x^n and sum from 2 to infinity to produce:

$$\sum_{n=2}^{\infty} a_n x^n = 3\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} x^n$$

Now adding $a_0 + a_1 x$ to both sides and using the initial conditions we have

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 3\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} x^n + 1 + x$$

which gives

$$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=0}^{\infty} x^n$$

or

$$G(x) - 3x^2 G(x) = \frac{1}{(1-x)}$$

or

$$G(x) = \frac{1}{(1-3x^2)}\frac{1}{(1-x)}$$

We leave it to the student to find the terms of the sequence represented by *G*.

