Section 7.2
Solving Linear Recurrence Relations

If

$$a_{g(n)} = f(a_{g(0)}, a_{g(1)}, \ldots, a_{g(n-1)})$$

find a closed form or an expression for $a_{g(n)}$.

Recall:

• *nth degree polynomials have n roots*:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

• *If the coefficients are real then the roots are real or occur in complex conjugate pairs.*

Recall the *quadratic formula*: If

$$ax^2 + bx + c = 0 \text{ then}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We assume you remember how to solve linear systems

$$Ax = b.$$

where A is an n x n matrix.

Solving recurrence relations can be very difficult unless the recurrence equation has a special form:

- $g(n) = n$ (single variable)
- the equation is *linear*:
  - sum of previous terms
  - no transcendental functions of the a_i's
  - no products of the a_i's
- constant coefficients: the coefficients in the sum of the a_i's are constants, independent of n.
- degree k: a_n is a function of only the previous k terms in the sequence
- homogeneous: If we put all the a_i's on one side of the equation and everything else on the right side, then the right side is 0.

Otherwise *inhomogeneous* or *nonhomogeneous*.

Examples:

- $a_n = (1.02)a_{n-1}$
  linear
  constant coefficients
  homogeneous
  degree 1

- $a_n = (1.02)a_{n-1} + 2^{n-1}$
  linear
  constant coefficients
  nonhomogeneous
  degree 1

- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$
  linear
  constant coefficients
  nonhomogeneous
  degree 3

- $a_n = ca_{n/m} + b$
  g does not have the right form

- $a_n = na_{n-1} + n^2 a_{n-2} + a_{n-1}a_{n-2}$
  nonlinear
  coefficients are not constants
  homogeneous
  degree 2

Solution Procedure

- **linear**
- **constant coefficients**
- **homogeneous**
- **degree k**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_{n-k} a_{n-k}$$

1. Put all a_i's on the left side of the equation, everything else on the right. If nonhomogeneous, stop (for now).

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \ldots - c_{n-k} a_{n-k} = 0$$

2. Assume a solution of the form $a_n = b^n$.

3. Substitute the solution into the equation, factor out the lowest power of b and eliminate it.

$$b^n - c_1 b^{n-1} - c_2 b^{n-2} - \ldots - c_{n-k} b^{n-k} = 0$$

$$b^{n-k}[b^k - c_1 b^{k-1} - \ldots - c_{n-k}] = 0$$

4. The remaining polynomial of degree k,

$$b^k - c_1 b^{k-1} - \ldots - c_{n-k}$$

is called the *characteristic polynomial*.

Find its k roots, $r_1, r_2, \ldots, r_k$.

5. If the roots are distinct, the general solution is

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

6. The coefficients $\alpha_1, \alpha_2, ..., \alpha_k$ are found by enforcing the initial conditions.

Solve the resulting linear system of equations:

$$
\begin{aligned}
a_0 &= \alpha_1 r_1^0 + \alpha_2 r_2^0 + ... + \alpha_k r_k^0 \\
a_1 &= \alpha_1 r_1^1 + \alpha_2 r_2^1 + ... + \alpha_k r_k^1 \\
&\vdots \\
a_{k-1} &= \alpha_1 r_1^{k-1} + \alpha_2 r_2^{k-1} + ... + \alpha_k r_k^{k-1}
\end{aligned}
$$

Example:

$$a_{n+2} = 3a_{n+1},\ a_0 = 4$$

• Bring subscripted variables to one side:

$$a_{n+2} - 3a_{n+1} = 0.$$

• Substitute $a_n = b^n$ and factor lowest power of b:

$$b^{n+1}(b-3) = 0 \text{ or } b-3 = 0$$

• Find the root of the characteristic polynomial:

$$r_1 = 3$$

• Compute the general solution:

$$a_n = c3^n$$

• Find the constants based on the initial conditions:

$$a_0 = c(3^0) \text{ or } c = 4$$

- Produce the specific solution:

$$a_n = 4(3^n)$$

Example:

$$a_n = 3a_{n-2},\ a_0 = a_1 = 1$$

- $a_n - 3a_{n-2} = 0$

Note: the a_{n-1} term has a coefficient of 0.

- $b^{n-2}(b^2 - 3) = 0$ or $b^2 - 3 = 0$
- $r_1 = \sqrt{3}, r_2 = -\sqrt{3}$
- $a_n = \alpha_1\sqrt{3}^n + \alpha_2(-\sqrt{3})^n$
- Solve the linear system for α_1, α_2:

$$a_0 = 1 = \alpha_1\sqrt{3}^0 + \alpha_2(-\sqrt{3})^0 = \alpha_1 + \alpha_2$$
$$a_1 = 1 = \alpha_1(\sqrt{3})^1 + \alpha_2(-\sqrt{3})^1 = \alpha_1\sqrt{3} - \alpha_2\sqrt{3}$$

Solve the first equation for the first variable and substitute in the second equation:

$$\alpha_1 = 1 - \alpha_2$$
$$1 = (1 - \alpha_2)\sqrt{3} - \alpha_2\sqrt{3} = \sqrt{3} - \alpha_2 2\sqrt{3}$$

$$\alpha_2 = (\sqrt{3} - 1) / 2\sqrt{3}$$
$$\alpha_1 = 1 - (\sqrt{3} - 1) / 2\sqrt{3} = (\sqrt{3} + 1) / 2\sqrt{3}$$

If a root r_1 has multiplicity p, then the solution is

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \ldots + \alpha_p n^{p-1} r_1^n + \ldots$$

Example:

$$a_n = 6a_{n-1} - 9a_{n-2}, a_0 = a_1 = 1$$

- Recurrence system:

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

- Find roots of characteristic polynomial

$$\begin{cases} b^2 - 6b + 9 = 0 \\ (b-3)^2 = 0 \end{cases}$$

- Roots are equal:

$$b_1 = b_2 = 3$$

- General solutions is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

• Solve for coefficients:

$$\begin{cases} a_0 = 1 = \alpha_1 + 0 \\ a_1 = 1 = 1(3^1) + \alpha_2 (1)(3^1) \\ \alpha_2 = -\dfrac{2}{3} \end{cases}$$

You finish.

Nonhomogeneous Recurrence Relations

- linear
- constant coefficients
- degree k

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_{n-k} a_{n-k} + f(n)$$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_{n-k} a_{n-k}$$

is the associated *homogeneous* recurrence equation

TELESCOPING

Note: we introduce the technique here because it will be useful to solve recurrence systems associated with divide and conquer algorithms later.

For recurrences which are

- first degree

$$a_n = \alpha a_{n-1} + f(n)$$

Method:

- back substitute

- force the coefficient of a_{n-k} on the left side to agree with the coefficient of a_{n-k} in the previous equation

- stop when we get to the initial condition on the right side

- add the left sides of the equations and the right sides of the equations and cancel like terms

- add the remaining terms together to get a formula for a_n.

Example:

- $a_n = 2a_{n-1} + 1, a_0 = 3$

- Write down the equation:

$$a_n = 2a_{n-1} + 1$$

• Write the equation for a_{n-1}:

$$a_{n-1} = 2a_{n-2} + 1$$

• Multiply by the constant which appears as a coefficient of a_{n-1} in the previous equation so the two will cancel when we add both sides:

$$2a_{n-1} = 2^2 a_{n-2} + 2$$

• Write down the equation for a_{n-2} and multiply both sides by the coefficient of a_{n-2} in the previous equation:

$$a_{n-2} = 2a_{n-3} + 1$$

becomes

$$2^2 a_{n-2} = 2^3 a_{n-3} + 2^2$$

• Continue until the initial condition appears on the right hand side:

$$a_1 = 2a_0 + 1$$

becomes

$$2^{n-1} a_1 = 2^n a_0 + 2^{n-1}$$

• Add both sides of the equations and cancel identical terms:

$$a_n = (2a_{n-1}) + 1$$

$$(2a_{n-1}) = [2^2 a_{n-2}] + 2$$

$$[2^2 a_{n-2}] = 2^3 a_{n-3} + 2^2$$

$$\vdots$$

$$2^{n-1}a_1 = 2^n a_0 + 2^{n-1}$$

$$a_n = 2^n a_0 + \sum_{i=0}^{n-1} 2^i$$

- Substitute a_0 and simplify $\sum_{i=0}^{n-1} 2^i$ to get the solution:

$$a_n = 3(2^n) + 2^n - 1 = 2^{n+2} - 1$$

Note: solution to nonhomogeneous case is sum of solution to associated homogeneous recurrence system and a particular solution to the nonhomogeneous case.

Theorem:

Let $\{a_n^P\}$ be a *particular* solution to the nonhomogeneous equation and let $\{a_n^H\}$ be the solution to the associated homogeneous recurrence system. Then every solution to the nonhomogeneous equation is of the form

$$\{a_n^H + a_n^P\}$$

Particular solution?

Theorem:

Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part *f(n)* of the form

$$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$$

If s is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form

$$(c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0) s^n$$

If s is a root of multiplicity m , a particular solutions is of the form

$$n^m (c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0) s^n$$

Example:

From the previous example the associated homogeneous recurrence equation is

$$a_n - 2a_{n-1} = 0$$

and

$$f(n) = 1$$

The root of the characteristic polynomial is 2 so the solution to the homogeneous part is

$$a_n^H = \alpha 2^n$$

and a particular solution to the nonhomogeneous equation is

$$\{a_n^P\} = c_0.$$

Substituting c_0 into the nonhomogeneous equation we get

$$c_0 - 2c_0 = 1$$

or

$$c_o = -1$$

Therefore the general solution is

$$\alpha 2^n - 1$$

Using the initial condition we have

$$\alpha 2^0 - 1 = 3 \text{ or } \alpha = 4 = 2^2$$

Hence, the solution is

$$a_n = 2^{n+2} - 1$$

which is the same solution we obtained by telescoping.

