Section 1.3
Predicates and Quantifiers

A generalization of propositions - *propositional functions* or *predicates*.: propositions which contain variables

Predicates become propositions once every variable is *bound*- by

• assigning it a value from the *Universe of Discourse* U

or

• quantifying it

Examples:

Let U = Z, the integers = {. . . -2, -1, 0 , 1, 2, 3, . . .}

• P(x): x > 0 is the predicate. It has no truth value until the variable x is bound.

Examples of propositions where x is assigned a value:

• P(-3) is false,
• P(0) is false,
• P(3) is true.

The collection of integers for which P(x) is true are the positive integers.

• $P(y) \quad \neg P(0)$ is not a proposition. The variable y has not been bound. However, $P(3) \quad \neg P(0)$ is a proposition which is true.

• Let R be the three-variable predicate R(x, y z): x + y = z

Find the truth value of

R(2, -1, 5), R(3, 4, 7), R(x, 3, z)

Quantifiers

- **Universal**

P(x) is true for every x in the universe of discourse.

Notation: *universal quantifier*

$$xP(x)$$

'For all x, P(x)', 'For every x, P(x)'

The variable x is bound by the universal quantifier producing a proposition.

Example: U={1,2,3}

$$xP(x) \Leftrightarrow P(1) \quad P(2) \quad P(3)$$

- **Existential**

P(x) is true for some x in the universe of discourse.

Notation: *existential quantifier*

$$xP(x)$$

‘There is an x such that P(x),’ ‘For some x, P(x)’, ‘For at least one x, P(x)’, ‘I can find an x such that P(x).’

Example: U={1,2,3}

$$xP(x) \Leftrightarrow P(1) \quad P(2) \quad P(3)$$

• **Unique Existential**

P(x) is true for one and only one x in the universe of discourse.

Notation: *unique existential quantifier*

$$!xP(x)$$

‘There is a unique x such that P(x),’ ‘There is one and only one x such that P(x),’ ‘One can find only one x such that P(x).’

Example:

U={1,2,3}

Truth Table:

P(1)	P(2)	P(3)	$!xP(x)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

How many minterms are in the DNF?

Note:

REMEMBER!

A predicate is <u>not</u> a proposition until *all* variables have been bound either by quantification or assignment of a value!

Equivalences involving the negation operator

$$\neg \quad xP(x) \Leftrightarrow \quad x\neg P(x)$$

$$\neg \quad xP(x) \Leftrightarrow \quad x\neg P(x)$$

Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

Multiple Quantifiers: read left to right . . .

Example: Let U = R, the real numbers,

P(x,y): xy= 0

$$x \quad yP(x,y)$$

$$x \quad yP(x,y)$$

$$x \quad yP(x,y)$$

$$x \quad yP(x,y)$$

The only one that is false is the first one.

Suppose P(x,y) is the predicate x/y=1?

Example:

Let U = {1,2,3}. Find an expression equivalent to

$$x \quad yP(x,y)$$

where the variables are bound by substitution instead:

Expand from inside out or outside in.

Outside in:

$$yP(1,y) \quad yP(2,y) \quad yP(3,y)$$
$$\Leftrightarrow [P(1,1) \quad P(1,2) \quad P(1,3)]$$
$$[P(2,1) \quad P(2,2) \quad P(2,3)]$$
$$[P(3,1) \quad P(3,2) \quad P(3,3)]$$

Converting from English

(can be very difficult)

Examples:

F(x): x is a fleegle
S(x): x is a snurd
T(x): x is a thingamabob

U={fleegles, snurds, thingamabobs}

(Note: the equivalent form using the existential quantifier is also given)

- Everything is a fleegle

$$xF(x)$$
$$\Leftrightarrow \neg \quad x\neg F(x)$$

- Nothing is a snurd.

$$x \neg S(x)$$
$$\Leftrightarrow \neg \quad xS(x)$$

- All fleegles are snurds.

$$x[F(x) \rightarrow S(x)]$$
$$\Leftrightarrow \quad x[\neg F(x) \quad S(x)]$$
$$\Leftrightarrow \quad x \neg [F(x) \quad \neg S(x)]$$
$$\Leftrightarrow \neg \quad x[F(x) \quad \neg S(x)]$$

- Some fleegles are thingamabobs.

$$x[F(x) \quad T(x)]$$
$$\Leftrightarrow \neg \quad x[\neg F(x) \quad \neg T(x)]$$

- No snurd is a thingamabob.

$$x[S(x) \rightarrow \neg T(x)]$$
$$\Leftrightarrow \neg \quad x[S(x) \quad T(x)]$$

- If any fleegle is a snurd then it's also a thingamabob

$$x[(F(x) \quad S(x)) \rightarrow T(x)]$$
$$\Leftrightarrow \neg \quad x[F(x) \quad S(x) \quad \neg T(x)]$$

Extra Definitions:

• An assertion involving predicates is *valid* if it is true for every universe of discourse.

• An assertion involving predicates is *satisfiable* if there is a universe and an interpretation for which the assertion is true. Else it is *unsatisfiable.*

• The *scope* of a quantifier is the part of an assertion in which variables are bound by the quantifier

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Examples:

Valid: $x\neg S(x) \leftrightarrow \neg \quad xS(x)$

Not valid but satisfiable: $x[F(x) \rightarrow T(x)]$

Not satisfiable: $x[F(x) \quad \neg F(x)]$

Scope: $x[F(x) \quad S(x)]$ vs. $x[F(x)] \qquad x[S(x)]$

Dangerous situations:

• Commutativity of quantifiers

$$x \quad yP(x,y) \Leftrightarrow \quad y \quad xP(x,y)?$$

YES!

$$x \quad yP(x,y) \Leftrightarrow \quad y \quad xP(x,y)?$$

NO!

DIFFERENT MEANING!

- Distributivity of quantifiers over operators

$$x[P(x) \quad Q(x)] \Leftrightarrow \quad xP(x) \qquad xQ(x)?$$

YES!

$$x[P(x) \rightarrow Q(x)] \Leftrightarrow [\quad xP(x) \rightarrow \quad xQ(x)]?$$

NO!

