Deduction/Inferencing in Predicate logic

- All axioms, laws, rules of inference of propositional logic with 0-order formula symbol viewed as 1^{st}-order formula symbol are axioms, laws, rules of inference of 1^{st}-order logic
- Recall 0-order formulae:
  - constants $= \{T, F\}$, and prop. variables $= \{p, q, r \ldots\}$ are formulae
  - α, β formulae $\Rightarrow \neg\alpha, \alpha \wedge \beta, \alpha \vee \beta$ formulae.
- Recall 1^{st}-order formulae:
  - constants $C = \{c_1, c_2, \ldots\}$, and variables $X = \{x_1, x_2, \ldots\}$ are terms
  - functions $F = \{f_1, f_2, \ldots\}$ generate more terms: $t_1, \ldots, t_n$ terms $\Rightarrow f(\vec{t})$ term
  - Relations $R = \{r_1, r_2, \ldots\}$ generate formulae: $t_1, \ldots, t_n$ terms $\Rightarrow r(\vec{t})$ formulae
  - $\alpha(\vec{x}), \beta(\vec{x})$ formulae $\Rightarrow \neg\alpha(\vec{x}), \alpha(\vec{x}) \wedge \beta(\vec{x}), \alpha(\vec{x}) \vee \beta(\vec{x}), \exists x\, \alpha(\vec{x}), \forall x\, \alpha(\vec{x})$

- Added rules of inference (for added operators, $\exists$ and $\forall$):

① $\exists$ introduction $\dfrac{\begin{matrix} a \in \mathcal{U} \\ f(a) \end{matrix}}{\exists x \in \mathcal{U}: f(x)}$ ② $\exists$ elimination $\dfrac{\exists x \in \mathcal{U}: f(x)}{a \in \mathcal{U} \wedge f(a)}$

③ $\forall$ elimination $\dfrac{\begin{matrix} a \in \mathcal{U} \\ \forall x\, f(x) \end{matrix}}{f(a)}$ ④ $\forall$ introduction $\dfrac{\begin{matrix} [a \in \mathcal{U}] \\ f(a) \end{matrix}}{\forall x \in \mathcal{U}: f(x)}$

- Example over number system : $\mathrm{Odd}(x_1) \wedge \mathrm{Odd}(x_2) \rightarrow \mathrm{Odd}(x_1 x_2)$

1.	$[\mathrm{Odd}(x_1) \wedge \mathrm{Odd}(x_2)]$	Assume
2.	$\mathrm{Odd}(x_i)$	$\wedge$-elimination
3.	$\exists y_i \in \mathbb{N} : x_i = 2y_i + 1$	"Odd" def.
4.	$x_i = 2y_i + 1$	$\exists$ elimination
5.	$(x_1 x_2) = (2y_1 + 1)(2y_2 + 1) = 2(2y_1 y_2 + y_1 + y_2) + 1$	"product" def
6.	$\exists z \in \mathbb{N} : x_1 x_2 = 2z + 1$	$\exists$ introduction
7.	$\mathrm{Odd}(x_1 x_2)$	"Odd" def
8.	$\mathrm{Odd}(x_1) \wedge \mathrm{Odd}(x_2) \rightarrow \mathrm{Odd}(x_1 x_2)$	$\rightarrow$ introduction

- Godel in 1930 (at age of 24) proved above 1^{st}-order logic (due to Hilbert) is sound & complete.