Reachability Analysis Based Minimal Load Shedding Determination

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Abstract—This paper proposes a method to determine the minimal amount of load shedding based on backward reachability analysis. The backward reachable set is calculated using level set methods to compute the region of stability of a stable equilibrium point. The minimal amount of load shedding as a function of time elapsed since the post-disturbance condition can be systematically derived as the horizontal distance between the post-disturbance trajectory and the stability boundary of the stable equilibrium point. Case studies with a system consisting of an infinite bus feeding a dynamic load through a tap-changing transformer are presented to illustrate the approach.

Index Terms—Level set methods, load shedding, reachability analysis, voltage stability.

I. INTRODUCTION

Voltage instability is one of the major threats to power system operation [1]. In this paper, we focus on long-term large-disturbance voltage stability. According to the recent report of the IEEE/CIGRE joint task force [2], long-term large-disturbance voltage stability is the ability of a power system involving slow acting equipment such as tap-changing transformers to maintain steady voltages at all buses after a large disturbance from a given initial operating condition.

In face of long-term large-disturbance voltage instability, load shedding is generally an effective corrective control candidate [3]. It disconnects a fraction of load in order to inhibit further system deterioration. It is desired that the percentage of load to be shed is minimized when designing an undervoltage load shedding scheme. The previous studies dealing with load shedding against voltage collapse can be classified into two categories: the static and the dynamic methods. The static methods [4, 5, 6, 7] concentrate on determining the minimal action needed in order to restore a solution to the power flow equations or to increase stability margins. However, in spite of their relative advantage in terms of computing time, none of the static methods are able to account for temporal influences. In particular the influence of the shedding delay, which has been shown in [8, 9] to be closely related to the minimal load to shed, is not taken into account. This problem requires models incorporating system dynamics [9, 10]. However, the minimal amount of load shedding as a function of time is usually obtained by simulation using trial and error [9] which is time consuming. In this paper, a systematic approach based on the region of stability is proposed to determine the minimal amount of load shedding as a function of time elapsed since the post-disturbance condition.

The region of stability of a stable equilibrium point refers to the region of operating conditions in state space that eventually tend to a stable equilibrium point. There have been several efforts made to estimate the region of stability [11]. Various methods using Lyapunov functions [12] are proposed to estimate the region of stability. In addition to the difficulty to construct “good” Lyapunov functions for general power system models, the methods based on Lyapunov functions can only find some subset of the exact region of stability, which is found to be overly conservative, with an unpredictably varying degree of conservativeness [13]. On the other hand, Chiang et al. in [14] analyzed the topological and dynamical characterization of the stability boundaries for a class of nonlinear autonomous dynamic systems. Under certain assumptions, the stability boundary of a stable equilibrium point was shown to consist of the stable manifolds of all the unstable equilibrium points on the stability boundary. The method is not applicable if the system only has one stable equilibrium point such as the reverse Van der Pol system [15]. What is more, finding the stable manifold of an equilibrium point is a nontrivial problem for third and higher order systems. In this paper, the exact region of stability of a stable equilibrium point is computed based on backward reachability analysis using level set methods, and illustrated for a system including a dynamic load and a tap-changing transformer. The proposed method does not require construction of Lyapunov functions. It is applicable for general nonlinear systems, and it does not require identification of the unstable equilibrium points.

The paper is organized as follows. Some fundamental concepts of the reachability analysis and level set methods are introduced in Section II. Section III describes an algorithm to determine the region of stability. In Section IV we calculate...
minimal load shedding based on the proposed method. Section V presents discussion and conclusions.

II. REACHABILITY ANALYSIS AND LEVEL SET METHOD

A. Reachability Analysis

We consider the following nonlinear autonomous system:

\[
\frac{dx}{dt} = f(x) \tag{1}
\]

where \( x \in \mathbb{R}^n \) is the state vector, and \( f(x) \) is the vector field. The stability problem in power system is mathematically formulated as one of ensuring that the state of the power system at the instant of clearing the fault is inside the region of stability (ROS) of the post-fault stable equilibrium point [11]. We use the backward reachable set analysis to obtain the ROS. Given system dynamics and a target set of states, the backward reachable set is the set of states from where \( \phi \) is replaced by \( \epsilon \)-ball around the stable equilibrium point. One way to describe a set of continuous states is known as the implicit surface function representation. Consider a closed set \( S \subseteq \mathbb{R}^n \). An implicit surface representation of \( S \) would define a function \( \phi(x): \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \phi(x) \leq 0 \) if \( x \in S \) and \( \phi(x) > 0 \) if \( x \notin S \). One way to track the backward reachable set of a target set is to solve the system equation (1) with \( t \) replaced by \(-t\) for all \( x \) on the boundary of the target set i.e., for all \( x \) with \( \phi(x,t=0)=0 \), where \( \phi(x,t) \) is the implicit surface representation of the backward reachable set at time \( t \), meaning the collection of states that can reach the target set in time \( t \) or less. Because \( \phi(x,t=0) = 0 \) can have infinitely many solutions, we discretize it into a finite number of subregions and calculate the evolution of all subregions simultaneously. However, the accuracy of the method can deteriorate quickly along the evolution of the implicit surface [16]. In order to avoid the above problems, we can use the implicit function \( \phi(x,t) \) both to represent the boundary of the backward reachable set and to evolve the boundary. Fig. 1 illustrates an implicit surface propagating with vector field \( f(x) \).

![Fig. 1. Implicit surface propagating with vector field \( f(x) \).](image)

The evolution of the implicit function \( \phi(x,t) \) is defined by [16]:

\[
\phi_t + (\nabla \phi)^T f(x) = 0, \tag{2}
\]

where the \( t \) subscript represents a partial derivative with respect to the time variable \( t \), \( \nabla \phi \) is the gradient of \( \phi(x) \). We can rewrite (2) as:

\[
\phi_t + \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} f_i(x) = 0. \tag{3}
\]

The partial differential equation (2) defines the motion of the boundary where \( \phi(x,t)=0 \). It is known as Elerian formulation of the interface evolution. Equation (2) is also referred to as the level set equation which was introduced by Osher and Sethian for numerical interface evolution of boundary [17].

B. Level Set Method

Actually, (2) is the special form of the following general Hamilton-Jacobi-Isaacs equation:

\[
\phi_t + H(\nabla \phi) = 0, \tag{4}
\]

where the Hamiltonian \( H \) can be a function of both space and time. For the autonomous nonlinear system given by (1),

\[
H(\nabla \phi) = (\nabla \phi)^T f(x). \tag{5}
\]

Unlike any other formulation of the backward reachable set, the HJI PDE formulation can be solved very accurately using numerical methods based on the level set techniques. Level set methods are a collection of numerical techniques that can track the implicit surfaces evolving according to the associated vector fields [18]. They have been used in a variety of image processing and computer vision, computational physics, fluid mechanics, and combustion application. In level set methods, a set of data points (known as a grid) is defined first. The most popular grids are Cartesian grids defined as:

\[
\{(x_i,y_j) : 1 \leq i \leq m, 1 \leq j \leq n\} \tag{6}
\]

Once \( \phi(x,0) \) and \( f(x) \) are defined at each Cartesian grid point, numerical methods can be applied to evolve \( \phi \) in time moving the boundary across the grid. If the region of stability is bounded, the sequence of evolving boundaries will eventually converge. Let \( \phi^n(x) = \phi(x,t_n) \) represent the value of \( \phi \) at time \( t_n \). In order to update \( \phi \) in time, we need to find new values of \( \phi \) at every grid point after some time increment \( \Delta t \). These new values of \( \phi \) are denoted as

\[
\phi^{n+1}(x) = \phi(x,t_{n+1}) \tag{7}
\]

where \( t_{n+1} = t_n + \Delta t \). We can discretize (2) as follows:

\[
\frac{\phi^{n+1}(x) - \phi^n(x)}{\Delta t} + (\nabla \phi^n)^T f(x) = 0 \tag{8}
\]

where \( \nabla \phi^n \) is the gradient of \( \phi^n \). Equation (9) can be rewritten as:
partial derivatives of $\phi^\epsilon$ with respect to $x_i$. One might evaluate these partial derivatives as:

$$\phi^\epsilon_i = \frac{\phi^\epsilon(x_i) - \phi^\epsilon(x_{i-1})}{\Delta x_i}$$

Higher-order accurate differencing methods such as the Hamilton-Jacobi nonoscillatory method [17] can also be used to increase the accuracy of the approximation for $\phi^\epsilon_i$.

### III. AN ALGORITHM TO DETERMINE REGION OF STABILITY

In this section we describe the algorithm to determine the region of stability of a SEP.

Step 1: Find a stable equilibrium point of an autonomous nonlinear system by solving $f(x)=0$, and let $x^* \in \mathbb{R}^n$ be a stable equilibrium point.

Step 2: Specify an $\epsilon$-ball centered at the stable equilibrium point with radius $\epsilon$ as the target set. Define an implicit surface function at time $t=0$ as

$$\phi(x,0) = \|x - x^*\| - \epsilon$$

Then the target set is the zero sublevel set of the function $\phi(x,0)$, i.e., it is given by,

$$\{x \in \mathbb{R}^n | \phi(x,0) \leq 0\} = \{x \in \mathbb{R}^n | \|x - x^*\| \leq \epsilon\}$$

Therefore, a point $x$ is inside the target set if $\phi(x,0)$ is negative, outside the target set if $\phi(x,0)$ is positive, and on the boundary of the target set if $\phi(x,0) = 0$.

Step 3: Propagate in time the boundary of the backward reachable set of the target set by solving the following HJI PDE:

$$\phi + \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} f_i(x) = 0$$

with terminal conditions

$$\phi(x,0) = \|x - x^*\| - \epsilon$$

The zero sublevel set of the viscosity solution $\phi(x,t)$ to (14), (15) is the backward reachable set at time $t$:

$$\{x \in \mathbb{R}^n | \phi(x,t) \leq 0\}$$

Step 4: The backward reachable set of the $\epsilon$-ball around the stable equilibrium point is computed using the toolbox of level set methods [19]. It is always contained in the region of stability of the stable equilibrium point. And if $t$ goes to infinity, the backward reachable set approaches the true region of stability. If the region of stability is bounded, the level-set methods based numerical computation of the backward reachability set eventually converges to the region of stability within a finite computation time.

### IV. APPLICATION TO MINIMAL LOAD SHEDDING

Because the dynamic load restoration mechanism plays an important role in voltage instability, we consider the system of Fig. 2 consisting of an infinite bus feeding a dynamic load through a lossless transmission line and an ideal tap-changing transformer. In the model, we use the continuous tap-changing transformer model [20] to approximate the discrete one for the purpose of a more convenient analytic analysis.

The system model is given as follows:

$$T_L G = P_s - GV_2^2$$

$$T_r r = V_2 - V_{\text{ref}}$$

$$(rV_2)^4 - E^2(rV_2)^2 + X^2(GV_2^2)^2 = 0$$

Equation (17) represents the dynamic behavior of a constant load where

- $V_2$ is load voltage
- $P_s$ is power set point, and in the context of load shedding, $P_s$ is the load power demand after the shedding action.
- $G$ is controllable variable load conductance which is adjusted to maintain constant power
- $T_L$ is load recovery time constant.

Note that we assume that the dynamic load has unity power factor.

Equation (18) is the approximated continuous tap-changing transformer model where

- $V_s$ is voltage set point
- $r$ is tap ratio of the tap-changing transformer
- $T_r$ is the time constant of the approximated continuous tap-changing transformer which has been derived in [21].

In Equation (19), $E$ is the voltage of the infinite bus, $X$ is the reactance of the transmission line.

From (19), we can get

$$V_2^2 = \frac{E^2 r^2}{r^4 + X^2 G^2}$$

Therefore, the set of equations (17)-(19) can be expressed as a set of first order ordinary differential equations (21)-(22):

$$T_L G = P_s - GV_2^2$$

$$T_r r = V_2 - V_{\text{ref}}$$

Equations (21) and (22) can be rewritten as:

$$G = \frac{1}{T_L} (P_s - GV_2^2) = f_1(G, r)$$

$$r = \frac{1}{T_r} (V_2 - V_s) = f_2(G, r)$$

The equilibrium conditions for the original model (17)-(19) are as follows:

$$V_2 = V_s$$
\[ G^* = \frac{P}{V_i^2} = G_s \]  
\[ (rV_s)^4 - E^2 (rV_s)^2 + X^2 P_s^2 = 0 \]  
(27)

Equation (27), for values of \( P_s \) lower than the maximum power given by

\[ P_{\text{max}} = \frac{E^2}{2X} \]  
(28)

has two solutions \( r_s^* \) and \( r_u^* \), where \( r_s^* \) corresponds to the stable and \( r_u^* \) to the unstable equilibrium point, and \( r_s^* > r_u^* \).

In the following example, the parameters are chosen as: \( X = 0.5, \ V_i = 1.0, \ P_s = 0.9, \ E = \sqrt{1.06} = 1.02956, \ T_L = 10 \text{ sec.}, \ T_\delta = 5 \text{ sec.} \) Then we get the stable equilibrium point as \( (G_s^* = 0.9, \ r_s^* = 0.9) \).  

A. Region of Stability

Following the algorithm of section III, we compute the region of stability of the stable equilibrium point \( (G_s^* = 0.9, \ r_s^* = 0.9) \) as follows:

1): Specify an \( \epsilon \)-ball centered at the stable equilibrium point \( (G_s^* = 0.9, \ r_s^* = 0.9) \) with radius \( \epsilon = 0.02 \) as the target set. The implicit surface function is defined as

\[ \phi((G, r), 0) = \sqrt{(G-0.9)^2 + (r-0.9)^2} - 0.02 \]  
(29)

Then the target set is the zero sublevel set of the function \( \phi((G, r), 0) \).

2): Calculate the backward reachable set of the target set by solving the following HJI PDE:

\[ \dot{\phi} + \frac{\partial \phi}{\partial G} (-f_G) + \frac{\partial \phi}{\partial r} (-f_r) = 0 \]  
(30)

with terminal conditions \( \phi((G, r), 0) \) given by (29).

For this example, the backward reachable set computation converges in 93.1 seconds.

Fig. 3 shows the computed region of stability of the stable equilibrium point of the post-disturbance system. In the figure, the stability boundary is indicated by the solid line. We have validated our results by plotting the phase portrait in the same figure computed using time domain simulation of sample trajectories. The dashed lines with arrows represent sample trajectories. It is clear that the computed region of stability is accurate.

We use state trajectories to verify our result. Fig. 4 and Fig. 5 show the state trajectories for two different initial conditions. From our computed region of stability, we conclude that the state trajectories starting from the initial condition \( (G_{(0)}=1.31, \ r_{(0)}=1.5) \) converge to the stable equilibrium point, whereas the state trajectories starting from the initial condition \( (G_{(0)}=1.32, \ r_{(0)}=1.5) \) diverge. Fig. 4 shows convergent trajectories and on the other hand, Fig. 5 shows that the voltage begins to collapse at about 35 seconds.

B. Minimal Load Shedding Determination

Once the region of stability of the stable equilibrium point is calculated, this information can be used to derive the minimal amount of load shedding as a function of the time elapsed since the initial post-disturbance condition required to stabilize the system. For example, when the initial condition of the post-disturbance system is \( (G_{(0)}=1.4, \ r_{(0)}=0.9) \), the trajectory is divergent which is indicated by the dashed curve in Fig. 6. Fig. 7 shows the trajectories with the initial condition \( (G_{(0)}=1.4, \ r_{(0)}=0.9) \), where we can see that the voltage begins to decrease at about 8 seconds. Let \( P_{(0)} \) be the load before shedding and \( G_{(0)} \) the corresponding admittance, such that the system is at equilibrium prior to the disturbance:
If a percentage \( \lambda \) of load is shed immediately after the disturbance, the post-shedding load conductance is:
\[
G = (1 - \lambda) G(0) \tag{32}
\]
The post-shedding load demand is:
\[
P = GV^2 \tag{33}
\]
If the post-shedding initial conditions are within the region of stability, the trajectories will converge to the stable equilibrium point. Therefore, the minimal amount of load shedding as a function of time elapsed since the initial post-disturbance condition can be derived as the horizontal distance between the post-disturbance trajectory and the stability boundary of the stable equilibrium point. Here, we use the horizontal distance because the tap ratio of the tap-changing transformer is unchanged at the instance of load shedding. For the chosen initial condition, the minimal amount of load conductance shedding (in. p.u.) as a function of time elapsed since post-disturbance condition is shown in Fig. 8.

Fig. 8 shows that the required minimal amount of load shedding increases dramatically after 10 seconds. The results are validated through time domain simulation for different time delay. Fig. 8 indicates that the minimal amount of load shedding at \( t=15.1 \) s is 0.23 p.u. It means that if the amount of load which is greater than or equal to 0.23 p.u. is shed at time 15.1 s, we can guarantee the system is stable. The time domain simulation shown in Fig. 9 confirms this result. On the other hand, if a less amount of load is shed at the time of 15.1 s, the system may lose stability. Fig. 10 gives the time domain simulation for the situation that 0.22 p.u. load is shed at the time of 15.1 s. It is evident from time domain simulation that the voltage collapses eventually. If the load shedding action is taken sooner, less load needs to be shed to stabilize the system. Fig. 8 shows that only 0.05 p.u. load needs to be shed at \( t=3.3 \) s in order to stabilize the system. The corresponding time domain simulation result is presented in Fig. 11.
This paper presents an approach for computing the minimal amount of load shedding required to stabilize a system that otherwise will diverge. The computation utilizes region of stability computation. The region of stability of a stable operating point of the post-disturbance system is computed as the backward reachable set of a small neighborhood of the operating point of the post-disturbance system is computed as the backward reachable set of a small neighborhood of the stable equilibrium point, and is computed using level set methods. The minimal amount of load shedding as a function of time elapsed since post-disturbance condition is systematically derived as the horizontal distance between the post-disturbance trajectory and the stability boundary of the stable operating point.

The numerical tests conducted on a model consisting of an infinite bus feeding a dynamic load through a tap-changing transformer show the effectiveness of the proposed approach. The presented method is applicable to more complex systems. The computation time of backward reachable sets is acceptable for systems with up to five state space dimensions. However, the method may not be efficient for larger power systems. Further work is needed to increase the efficiency of the backward reachable set computation algorithm.

VI. REFERENCES


VII. BIOGRAPHIES

Haifeng Liu received the B.S. and M.S. degrees in electrical engineering from Zhejiang University, Hangzhou, China, in 2000 and 2003, respectively. He is currently pursuing the Ph.D. degree in the Department of Electrical and Computer Engineering at Iowa State University, Ames. His research interest is power system hybrid control.

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