Digital Filters

- Analog signals can be discretized and filtered in discrete time to allow flexibility of filtering in "software" (as opposed to "hardware").

- Discretizing time is not enough; amplitude must also be discretized. Here we only discuss time-domain discretization; error introduced due to discretization of amplitude, called quantization error, can be computed.

- Two types of discrete-time filters
  - FIR (finite impulse response): $h[k] = 0$ for finitely many $k$  
    $\Rightarrow H(z)$ a polynomial in $z^{-1}$; BIBO stable.  
    Has finite memory, and so transients exist for finite duration.  
    Can approximate a desired mag. response without phase distortion (with linear phase).
  - IIR (infinite impulse response): $h[k] \neq 0$ for infinitely many $k$.  
    $H(z)$ a rational function (ratio of two polynomials) in $z^{-1}$; stability an issue.  
    Effects of transients not limited to finite duration since infinite memory  
    can compactly approximate a desired mag. response, but with phase distortion  
    (phase shift linear in freq. of freq. $\Rightarrow$ All freqs. incur equal delay).

FIR Filters

Desired freq. response: $H_d(e^{j\omega})$; freq. response of FIR approx. $H(e^{j\omega})$.  

The mean sq. error given by:  
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 d\omega,$$

should be minimized. This equals  
$$\sum_{n=-M}^{M} (h_d[n] - h[n])^2 \quad ( Parsecde\ Thm )$$

For FIR design, $h[n] = \left\{ \begin{array}{ll} h_d[n] & 0 \leq n \leq M \quad (M: \text{order of filter}) \\ 0 & \text{otherwise} \end{array} \right.$$

Equivalently,  
$$h[n] = h_d[n] \cdot [n] \quad \text{with} \quad [n] = \left\{ \begin{array}{ll} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{array} \right.$$

Here $[n]$ is a "rectangular window".
FIR Filter

Rectangular window, \( w[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \)

Freq. response, \( W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} \)

\[
\begin{align*}
W(e^{j\omega}) &= \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\omega M/2} \frac{\sin (\omega (M+1)/2)}{\sin (\omega/2)}}{e^{-j\omega M/2}} \\
&= e^{j\omega M/2} \frac{\sin (\omega (M+1)/2)}{\sin (\omega/2)} \\
&= \left(\frac{\sin (\omega (M+1)/2)}{\sin (\omega/2)}\right) (\omega \leq \pi)
\end{align*}
\]

Width of main-lobe, \( \Delta \omega_{\text{mainlobe}} = \frac{4\pi}{M+1} \)

Width of side-lobes = \( 2\pi/\text{M+1} \)

For large \( M \), \( \frac{1}{3}\sin (\frac{2\pi}{M+1}) \approx \frac{2\pi}{3\pi} \Rightarrow \) peak of main-lobe \( \approx \frac{3\pi}{2} \approx 13\text{dB} \)

Another choice for window is Hamming window:

\( w[n] = \begin{cases} 0.54 - 0.46 \cos \left( \frac{2\pi n}{M} \right) & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \)

Figure 3.18 of textbook (also see next slide) compares amplitude responses of rectangular and Hamming windows.

- Mainlobe of Hamming window is more than double in width than main-lobe of rectangular window.

- Side-lobes of Hamming window, relative to main-lobe, are greatly reduced compared with those of rectangular window (40dB vs. 13dB)

See Fig 3.18 (also next slide)
Figure 8.18 (p. 638)
Comparison of magnitude responses of rectangular and Hamming windows for a filter of order $M = 12$, plotted in decibels.
Start with a desired frequency response, and introduce a linear phase-shift with slope $M/2$ so that the corresponding impulse response is delayed by $M/2$, where the window peaks. Take inverse FT to obtain the desired impulse response, and truncate by a suitable window to make the impulse response finite.

**Example: Low-pass filter**

$$h_d(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & |\omega| \leq Mc \\ 0 & Mc < |\omega| < \pi \end{cases}$$

$$\Rightarrow h_d[n] = \frac{1}{2\pi} \int_{-Mc}^{Mc} (e^{-j\omega M/2}) e^{j2\pi n\omega} d\omega = \frac{1}{2\pi} \int_{-Mc}^{Mc} e^{j2\pi n\omega} (e^{-j\omega M/2}) d\omega$$

$$= \frac{1}{2\pi} \left[ \frac{e^{j2\pi nM/2} - e^{-j2\pi nM/2}}{j2\pi n} \right] = \frac{\sin [j2\pi n(M/2)]}{\pi (n - M/2)}$$

$$= \frac{M}{\pi} \frac{\sin [2\pi n(M/2)]}{2\pi n(M/2)} \quad -\infty < n < +\infty$$

Note: Maximum for $h_d[n]$ occurs at $n = M/2$, and $h_d[M/2] = \frac{M}{\pi}$.

So, $h[n] = h_d[n] w[n] = \begin{cases} \frac{M}{\pi} \frac{\sin [2\pi n(M/2)]}{2\pi n(M/2)} & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$

**Hamming window**

$$= \begin{cases} \frac{M}{\pi} \frac{\sin [2\pi n(M/2)]}{2\pi n(M/2)} (0.54 - 0.46 \cos(2\pi n/M)) & 0 < n < M \\ 0 & \text{otherwise} \end{cases}$$

**Speech signal processing example:** Fig 8.22 "This was easy for us" in female voice. Without/with LPF. Fig 8.23 corresponding spectra.

Sampling freq. = 16 kHz = Sampling period = 62.5 msec.

#samples = 27, 751

Cut-off freq. for LPF = 31 kHz, order of filter = 98, $\alpha = 3.14 \times 62.5 \times 10^3 \times 2\pi$. 
Figure 8.19 (p. 639)
Comparison of magnitude responses (plotted on a dB scale) of two low-pass FIR digital filters of order $M = 12$ each, one filter using the rectangular window and the other using the Hamming window.
Figure 8.22 (p. 643)
(a) Waveform of raw speech signal, containing an abundance of high-frequency noise. (b) Waveform of speech signal after passing it through a low-pass FIR digital filter of order $M = 98$ and cutoff frequency $f_c = 3.1 \times 10^3$ rad/s.
Figure 8.23 (p. 644)
(a) Magnitude spectrum of unfiltered speech signal. (b) Magnitude spectrum of filtered speech signal. Note the sharp cutoff of the spectrum around 3100 Hz.