ECE570
Engineering Mathematics
Homework # 3

Problem 1 Let $x_1 = [1\ 0\ 1]'$, $x_2 = [1\ 1\ 0]'$, $x_3 = [1\ 2\ 3]'$. In the inner-product defined by $< x, y >= y'x$ orthogonalize the above three vectors using the Gram-Schmidt procedure. Orthogonalize them using a new inner-product $< x, y >_n = x_1 y_1 + 2x_2 y_2 + x_3 y_3$.

problem 2 Show that any plane rotation matrix is orthogonal

Problem 3 Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Show that $null(A) = range(A) = span([1\ 0]')$.

Problem 4 Let V be the vector space of real column 2-vectors with and inner product defined by $< x, y >= x_1 y_1 + 2x_2 y_2$. Let $\mathcal{A} : V \to V$ be a linear operator which has a matrix representation $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the basis e_1 and e_2. Find the matrix representation of the adjoint of the operator $\mathcal{A}$ in the basis e_1 and e_2 and ascertain if it is self-adjoint.

Problem 5 The following are norms on R^n

$$\|x\|_\infty := \max_{1 \le n} |x_i|, \ \|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2) \right)^{\frac{1}{2}}, \quad \text{and } \|x\|_1 := \sum_{i=1}^{n} |x_i|$$

that are called the ℓ_∞, ℓ_2 and ℓ_1 norms respectively. Prove that

1. $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$.
2. $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$
3. $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$

Problem 6 Prove that if $\|.\|_n : R^n \to R$ is any norm on R^n then

$$||A||_{ind} := \max_{\|x\|_n = 1} \|Ax\|_m = \max_{\|x\|_n \le 1} \|Ax\|_m = \max_{x \ne 0} \frac{\|Ax\|_m}{\|x\|_n}$$

where $A : R^n \to R^m$ is a matrix.

Problem 7 For any induced-norm of a matrix as defined in Problem 6 prove the following

1. $\|A\|_{ind} \geq 0$
2. $\|A\|_{ind} = 0$ implies $A = 0$.
3. $\|\alpha A\|_{ind} = |\alpha|\|A\|_{ind}$ for all scalars α.
4. $\|A + B\|_{ind} \leq \|A\|_{ind} + \|B\|_{ind}$.
5. $\|AB\|_{ind} \leq \|A\|_{ind}\|B\|_{ind}$.

Problem 8 The spectral radius of a $n \times n$ matrix A is defined as

$$\rho(A) := \max_{1 \leq i \leq n} |\lambda_i|.$$

Prove the following

1. $\rho(A^*A)^{\frac{1}{2}} = \|A\|_{2-ind} := \max_{\|x\|_2=1} \|Ax\|$.
2. $\|A\|_{ind} \geq \rho(A)$ where $\|A\|_{ind}$ is any induced norm of the matrix A.

Problem 9 Prove the following theorem: *An $n \times n$ matrix A is norm preserving in the ℓ_2 norm if and only if it is unitary.*

Problem 10 Verify the polarization formula:

$$< x, y >= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

where $\|.\|$ is a norm induced from the inner product $< ., . >$.

Problem 11 Prove Schur's theorem: *If A is a $n \times n$ matrix, then there is a unitary matrix P such that $P^*AP = T$, where T is upper triangular.*

Problem 12 Let T be normal upper triangular matrix. Show that the matrix T is diagonal.

Problem 13 Let $A := ww^*$ where w is a complex $n - vector$. Show that

1. A is Hermitian.
2. A has one eigenvalue w^*w.
3. A has $n - 1$ eigenvalues at 0.
4. $Span(w)^{\perp}$ is an $n-1$ dimensional A invariant subspace spanned by eigenvectors of A.

Problem 14 Let A be a $n \times n$ Hermitian matrix. Show that

1. If A is nonsingular, then A^{-1} is Hermitian.
2. If A is positive-definite then A^{-1} is positive-definite.
3. If A is nonsingular and indefinite then A^{-1} is indefinite.

Problem 15 Let $A = P\Lambda P^*$ be a Hermitian positive-definite matrix and $\Lambda := diag(\lambda_1, \ldots, \lambda_n)$. Define $D := diag(\lambda_1^{\frac{1}{2}}, \ldots, \lambda_n^{\frac{1}{2}})$. Show that $A^{\frac{1}{2}} := PD^{\frac{1}{2}}P^*$ defines an unique square root of the matrix A.

Problem 16 Find the Jordan Canonical form and the singular-value decompositions of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

Problem 17 A $n \times n$ matrix is said to be nilpotent if $A^p = 0$ for some positive integer p. Show that if any matrix is strictly upper or lower triangular then it is nilpotent. Show also that a matrix is nilpotent if and only if all its eigenvalues are zero.

Problem 18 Let $A := uv^*$ be a rank 1 matrix with u and v being complex n vectors. Find the Jordan-canonical form of A.

Problem 19 Let A be a nonsingular matrix. Use the Cayley-Hamilton theorem to write down A^{-1} as a polynomial in A.

Problem 20

1. Let A be Hermitian and positive semi-definite. Show that the SVD of A is $P\Lambda P^*$ where Λ is diagonal. Compare this result with the unitarily similar to diagonal form of Hermitian matrices introduced in class.
2. Let A be a $m \times n$ matrix and let P and Q be unitary matrices. Show that PAQ has the same singular values as A.

Problem 21 Prove that $| < x, y > | = \|x\|\|y\|$ only if either $y = 0$ or $y = \lambda x$ for some scalar λ.

Problem 22 Complete the proof of the Gram-Schmidt orthnormalization method by proving $span\{x_1, \ldots, x_n\} = span\{e_1, \ldots, e_n\}$ (see class notes for the theorem statement).