The first parametric subspace-based method was the *Pisarenko method*, which was further modified, leading to the MUltiple SIgnal Classification (MUSIC) method.

**MUSIC Method:** Recall the model

\[
\mathbf{x}(t) = A\mathbf{s}(t) + \mathbf{n}(t),
\]

where \( A \) is a Vandermonde matrix

\[
A = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_L} \\
\vdots & \vdots & \ddots & \vdots \\
e^{-j\omega_1(N-1)} & e^{-j\omega_2(N-1)} & \cdots & e^{-j\omega_L(N-1)}
\end{bmatrix}.
\]

\[
R = E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = AE\{\mathbf{s}(t)\mathbf{s}^H(t)\}A^H + \sigma^2 I = ASA^H + \sigma^2 I,
\]

and \( S = E\{\mathbf{s}(t)\mathbf{s}^H(t)\} \) is a full-rank diagonal matrix.
From the eigenanalysis of $R$, we see that

- the eigenvalues of $R$ are:
  
  $\lambda_k > \sigma^2, \quad k = 1, \ldots, L \quad \text{signal subspace}$
  
  $\lambda_k = \sigma^2, \quad k = L + 1, \ldots, N \quad \text{noise subspace}$,

- the noise subspace eigenvectors are orthogonal to the column space of $A$.

Let the signal and noise subspace eigenvectors be given by

$E = \begin{bmatrix} u_1, u_2, \ldots, u_L \end{bmatrix} \quad \text{signal subspace,}$

$G = \begin{bmatrix} u_{L+1}, u_{L+2}, \ldots, u_N \end{bmatrix} \quad \text{noise subspace}.$

Then

$$RG = \sigma^2 G.$$ 

On the other hand

$$RG = (ASA^H + \sigma^2 I)G = ASA^H G + \sigma^2 G.$$ 

Comparing the above two equations, we conclude

$$A^H G = 0.$$
The noise subspace eigenvectors of $R$ are orthogonal to the columns of $A$. In turn, the signal-subspace eigenvectors span the same subspace as the column space of $A$.

The true frequencies $\{\omega\}_{l=1}^L$ are the solutions to

$$\mathbf{a}^H(\omega)GG^H\mathbf{a}(\omega) = N - \mathbf{a}^H(\omega)EE^H\mathbf{a}(\omega) = 0$$

where

$$GG^H = P_A^\perp, \quad EE^H = I - GG^H = P_A.$$

*MUSIC* Spectral Estimate:

$$P_{\text{MUSIC}}(\omega) = \frac{1}{\mathbf{a}^H(\omega)GG^H\mathbf{a}(\omega)} = \frac{1}{N - \mathbf{a}^H(\omega)EE^H\mathbf{a}(\omega)}.$$

In practice, we do not know $R$, so:

$$\hat{P}_{\text{MUSIC}}(\omega) = \frac{1}{\mathbf{a}^H(\omega)\hat{G}\hat{G}^H\mathbf{a}(\omega)} = \frac{1}{N - \mathbf{a}^H(\omega)\hat{E}\hat{E}^H\mathbf{a}(\omega)}.$$

where

$$\hat{R} = \frac{1}{K} \sum_{k=1}^K \mathbf{x}(k)\mathbf{x}^H(k),$$

$$\hat{G}\hat{G}^H = P_A^\perp, \quad \hat{E}\hat{E}^H = I - \hat{G}\hat{G}^H = \hat{P}_A.$$
Remarks:

- the number of signals $L$ must be known or estimated,
- MUSIC involves eigendecomposition,
- computational cost can be quite intensive if the MUSIC estimate is evaluated with a fine grid.
Root-MUSIC Algorithm

Instead of computing the spectral MUSIC estimate, root the polynomial

\[ a^T(1/z) \hat{G} \hat{G}^H a(z) = 0, \]

where

\[ a(z) = \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N+1} \end{bmatrix}. \]

Remarks:

- MUSIC polynomial has the order \( 2N - 2 \), and, therefore, \( 2N - 2 \) roots,

- the roots form \( N - 1 \) pairs, where one root is the conjugate reciprocal of another, i.e. if \( z \) is a root, then \( 1/z^* \) will be a root as well.

\[ 0 = a^T(1/z) \hat{G} \hat{G}^H a(z) = [a^T(1/z) \hat{G} \hat{G}^H a(z)]^H \]
\[ = a^T(z^*) \hat{G} \hat{G}^H a(1/z^*) \]
\[ = a^T(1/\tilde{z}) \hat{G} \hat{G}^H a(\tilde{z}) \] with \( \tilde{z} = \frac{1}{z^*} \).

If \( z \) \equiv \text{root}, then \( 1/z^* \equiv \text{root too!} \)
Example: Let $z = 0.8 e^{j\pi/4}$ be a root of the MUSIC polynomial. Then, the conjugate-reciprocal root is

$$\frac{1}{z^*} = \frac{1}{0.8e^{-j\pi/4}} = 1.25 e^{j\pi/4}.$$ 

Thus, the angle of the root does not change but it lies at the opposite side of the unit circle.

Select only the roots that lie inside the unit circle. Estimates of signal frequencies:

$$\hat{\omega} = \angle \{z_l\}, \quad l = 1, 2, \ldots, L$$

using $L$ roots closest to the unit circle (so-called signal roots) with $|z| \leq 1$. 
Root-MUSIC has better performance at low SNR or small number of samples than spectral MUSIC because it is insensitive to radial errors (more precisely, sensitive only to errors that cause subspace swapping).

Root-MUSIC has much simpler implementation than spectral MUSIC because it does not require any search over frequency.
Minimum-norm Method

\[ \hat{P}_{MN}(\omega) = \frac{1}{|\alpha^H(\omega)w|^2}, \]

where the vector \( w \)

- has the first element equal to 1 and minimum norm,
- \( w \) belongs to the sample noise subspace.

Optimization problem:

\[
\min_w w^H w \quad \text{subject to} \quad w^H e_1 = 1, \quad \hat{G}\hat{G}^H w = w.
\]

Substituting the second constraint into the objective function and first constraint yields:

\[
w^H w = w^H \hat{G}\hat{G}^H \hat{G}\hat{G}^H w = w^H \hat{G}\hat{G}^H w,
\]

\[
w^H e_1 = w^H \hat{G}\hat{G}^H e_1 = 1.
\]

With these expressions, our optimization problem transforms to

\[
\min_w w^H \hat{G}\hat{G}^H w \quad \text{subject to} \quad w^H \hat{G}\hat{G}^H e_1 = 1.
\]
Hence,

\[
Q(w) = w^H \hat{G}\hat{G}^H w + \lambda (1 - w^H \hat{G}\hat{G}^H e_1) + \lambda^* (1 - e_1^H \hat{G}\hat{G}^H w)
\]

\[
(\nabla Q)^* = \hat{G}\hat{G}^H w - \lambda \hat{G}\hat{G}^H e_1 \implies \hat{G}\hat{G}^H w = \lambda \hat{G}\hat{G}^H e_1.
\]

Substituting the solution \(\hat{G}\hat{G}^H w = \lambda \hat{G}\hat{G}^H e_1\) to the constraint equation \(w^H \hat{G}\hat{G}^H e_1 = 1\), we get

\[
\lambda^* e_1^H \hat{G}\hat{G}^H e_1 = 1 \implies \lambda = \frac{1}{e_1^H \hat{G}\hat{G}^H e_1}.
\]

Finally,

\[
w = \lambda \hat{G}\hat{G}^H e_1 = \frac{1}{e_1^H \hat{G}\hat{G}^H e_1} \hat{G}\hat{G}^H e_1,
\]

where we used the second constraint. Substituting this solution into the expression for \(\hat{P}_{MN}(\omega)\), we get

\[
\hat{P}_{MN}(\omega) = \frac{1}{|a^H(\omega)w|^2} = \frac{(e_1^H \hat{G}\hat{G}^H e_1)^2}{|a^H(\omega)\hat{G}\hat{G}^H e_1|^2}.
\]

The constant in the numerator does not alter the shape of the spectrum and, as a rule, is omitted:

\[
\hat{P}_{MN}(\omega) = \frac{1}{|a^H(\omega)\hat{G}\hat{G}^H e_1|^2} = \frac{1}{|1 - a^H(\omega)\hat{E}\hat{E}^H e_1|^2}.
\]
ESPRIT Method

Consider

\[ A = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\ e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_L} \\
\vdots & \vdots & \ddots & \vdots \\
e^{-j\omega_1(N-1)} & e^{-j\omega_2(N-1)} & \cdots & e^{-j\omega_L(N-1)}
\end{bmatrix}. \]

Let \( \overline{A} \) and \( \underline{A} \) be the matrices with eliminated first and last row, respectively.

It can be readily shown that

\[ \overline{A}D = \underline{A}, \]

\[ D = \text{diag}\{e^{j\omega_1}, \ldots, e^{j\omega_L}\}. \]

Let \( \overline{E} \) and \( \underline{E} \) be formed from the signal eigenvector matrix \( E \) in the same way as \( \overline{A} \) and \( \underline{A} \) from \( A \).
Recall that $E$ and $A$ span the same (signal) subspace. Therefore

\[ E = AC \quad \implies \quad E = AC = \overline{AD}C, \quad \overline{E} = \overline{AC} \implies \]

\[ \overline{EC}^{-1}D^{-1}C = \overline{A}DCC^{-1}D^{-1}C = \overline{AC} = \overline{E}. \]

Equivalently

\[ \overline{E} = \overline{EC}^{-1}DC \quad \implies \quad \overline{E} = \overline{E}\Psi \]

where

\[ \Psi = C^{-1}DC. \quad (\ast) \]

In practice, both $C$ and $D$ are unknown!

LS solution for $\Psi$:

\[ \Psi = (\overline{E}^H\overline{E})^{-1}\overline{E}^H\overline{E}. \]

From $(\ast)$ it follows that the diagonal elements of $D$ are the eigenvalues of $\Psi$!
ESPRIT:

**Step 1:** Compute the eigendecomposition of the sample covariance matrix $\hat{R}$ and obtain the sample signal subspace $\hat{E}$.

**Step 2:** Form the matrices $\hat{E}$ and $\hat{E}$.

**Step 3:** Compute

$$\hat{\Psi} = (\hat{E}^H \hat{E})^{-1} \hat{E}^H \hat{E}.$$ 

**Step 4:** Form the eigenvalues $\hat{\psi}_l, l = 1, 2, \ldots, L$ of $\hat{\Psi}$ and obtain the frequency estimates as follows:

$$\hat{\omega}_l = \angle \hat{\psi}_l.$$
Model-fitting-based Parametric Spectral Analysis

**Nonlinear LS:** Recall low-rank modeling and obtain the frequencies by minimizing

\[
\min_{S, \omega} \left\{ \sum_{t=1}^{K} \| x(t) - A(\omega) s(t) \|^2 \right\} = \min_{S, \omega} \| X - A(\omega) S \|^2 \implies
\]

\[
\min_{\omega} \text{tr} \{ P_A(\omega) \hat{R} \} \iff \max_{\omega} \text{tr} \{ P_A(\omega) \hat{R} \}.
\]
Nonparametric and Parametric Methods: Relationship

Matrix Inversion Lemma:

\[(H + BCD)^{-1} = H^{-1} - H^{-1}B(C^{-1} + DH^{-1}B)^{-1}DH^{-1}\]

for arbitrary square nonsingular \(H\) and \(C\).

Consider the familiar expression for the covariance matrix

\[R = ASA^H + \sigma^2 I\]

and apply matrix inversion lemma to obtain

\[R^{-1} = (\sigma^2 I + ASA^H)^{-1} = \frac{1}{\sigma^2} (I + \frac{1}{\sigma^2} ASA^H)^{-1}\]

\[= \frac{1}{\sigma^2}[I - A(\sigma^2 S^{-1} + A^H A)^{-1} A^H],\]

which implies

\[\lim_{\sigma^2 \to 0} \{\sigma^2 R^{-1}\} = \lim_{\sigma^2 \to 0} \{I - A(\sigma^2 S^{-1} + A^H A)^{-1} A^H\} \]

\[= I - A(A^H A)^{-1} A^H = P_A^\perp.\]

Compare Capon and MUSIC spectra:

\[P_{\text{CAPON}}(\omega) = \frac{1}{a^H(\omega) R^{-1} a(\omega)}, \quad P_{\text{MUSIC}}(\omega) = \frac{1}{a^H(\omega) P_A^\perp a(\omega)}\]
as well as AR (max entropy) and min-norm spectra:

\[ P_{AR}(\omega) = \frac{1}{|a^H(\omega)R^{-1}e_1|^2}, \quad P_{MN}(\omega) = \frac{1}{|a^H(\omega)P_A^\perp e_1|^2}. \]

Clearly, for high SNR \((\sigma \to 0)\), we obtain

\[ P_{\text{CAPON}}(\omega) \sim P_{\text{MUSIC}}(\omega), \]
\[ P_{AR}(\omega) \sim P_{MN}(\omega). \]
Matlab Example

- 3 equi-power ($A_1 = A_2 = A_3$) complex exponentials with frequencies $f_1 = 0.1$, $f_2 = 0.15$, and $f_3 = 0.4$,

- zero-mean unit-variance complex Gaussian noise

- SNR and number of samples used to estimate the covariance matrix varied.
Blackman-Tukey Spectral Estimate, 
SNR = 100 dB, 100 Samples
Capon Spectral Estimate,
SNR = 100 dB, 100 Samples
Maximum-entropy (AR) Spectral Estimate, 
$\text{SNR} = 100 \text{ dB}, 100 \text{ Samples}$
MUSIC Spectral Estimate, 
SNR = 100 dB, 100 Samples
Minimum-norm Spectral Estimate, 
\[ \text{SNR} = 100 \text{ dB}, \ 100 \text{ Samples} \]
Blackman-Tukey Spectral Estimate,
$\text{SNR} = 10 \text{ dB, 100 Samples}$
Capon Spectral Estimate,
SNR = 10 dB, 100 Samples
Maximum-entropy (AR) Spectral Estimate,

$\text{SNR} = 10 \text{ dB}, 100 \text{ Samples}$
MUSIC Spectral Estimate,
$\text{SNR} = 10 \ \text{dB}, \ 100 \ \text{Samples}$
Minimum-norm Spectral Estimate,
SNR = 10 dB, 100 Samples
Blackman-Tukey Spectral Estimate,

\( \text{SNR} = 0 \ \text{dB}, \ 100 \ \text{Samples} \)
Capon Spectral Estimate,
SNR = 0 dB, 100 Samples
Maximum-entropy (AR) Spectral Estimate,
$\text{SNR} = 0 \text{ dB}, \ 100 \ \text{Samples}$
MUSIC Spectral Estimate,
SNR = 0 dB, 100 Samples
Minimum-norm Spectral Estimate, \( SNR = 0 \) dB, 100 Samples
Blackman-Tukey Spectral Estimate,
\[ \text{SNR} = -10 \text{ dB}, \ 10000 \text{ Samples} \]
Capon Spectral Estimate,
SNR = $-10$ dB, 10000 Samples
Maximum-entropy (AR) Spectral Estimate,
$\text{SNR} = -10 \text{ dB}, \ 10000 \text{ Samples}
MUSIC Spectral Estimate,
$\text{SNR} = -10 \text{ dB}, \ 10000 \text{ Samples}$
Minimum-norm Spectral Estimate,
SNR = $-10 \text{ dB}$, 10000 Samples