Frequency-Domain Analysis

Fourier Series

Consider a continuous complex signal

\[ x(t) \in [-T/2, T/2]. \]

Represent \( x(t) \) using an arbitrary orthonormal basis \( \varphi_n(t) \):

\[ x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n(t) \]

Orthonormality condition:

\[ \frac{1}{T} \int_{-T/2}^{T/2} \varphi_n(t) \varphi^*_k(t) dt = \delta(n - k). \]

Multiplying the above expansion with \( \varphi^*_k(t) \) and integrating over the interval, we obtain

\[ \frac{1}{T} \int_{-T/2}^{T/2} x(t) \varphi^*_k(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n(t) \varphi^*_k(t) dt \]

\[ = \sum_{n=-\infty}^{\infty} \alpha_n \left( \frac{1}{T} \int_{-T/2}^{T/2} \varphi_n(t) \varphi^*_k(t) dt \right) \]

\[ = \sum_{n=-\infty}^{\infty} \alpha_n \delta(n - k) = \alpha_k. \]
Thus, the coefficients of expansion are given by

\[ \alpha_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \varphi_k^*(t) dt. \]

**Proposition.** The functions \( \varphi_n(t) = \exp(j2\pi nt/T) \) are orthonormal at the interval \([-T/2, T/2]\).

**Proof.**

\[
\frac{1}{T} \int_{-T/2}^{T/2} \varphi_n(t) \varphi_k^*(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} e^{j2\pi(n-k)t} dt
\]

\[
= \frac{\sin[\pi(n-k)]}{\pi(n-k)} = \delta(n-k).
\]

Thus, we can take exponential functions \( \varphi_n(t) = \exp(j2\pi nt/T) \) as orthonormal basis \( \implies \) we obtain Fourier series.

**Fourier series for a periodic signal** \( x(t) = x(t + T) \):

\[
x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n}{T} t}
\]

\[
X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n}{T} t} dt.
\]
Fourier coefficients can be viewed as a signal spectrum:

\[ X_n \sim X(\Omega_n), \quad \text{where} \quad \Omega_n = \frac{2\pi n}{T} \Rightarrow \]

Fourier series can be applied to analyze signal spectrum! Also, this interpretation implies that periodic signals have discrete spectrum.

**Example:** Periodic sequence of rectangles:

\[
X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n t}{T}} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} A e^{-j\frac{2\pi n t}{T}} dt
\]

\[
= \frac{A\tau}{T} \cdot \frac{\sin(\pi n \frac{\tau}{T})}{\pi n \frac{\tau}{T}} \quad \text{real coefficients.}
\]
Remarks:

- In general, Fourier coefficients are complex-valued,

- For real signals, $X_{-n} = X_n^*$.

- Alternative expressions exist for trigonometric Fourier series, exploiting summation of sine and cosine functions.
Convergence of Fourier Series

Dirichlet conditions:

**Condition 1.** $x(t)$ is *absolutely integrable* over one period, i.e.

\[ \int_{T} |x(t)| dt < \infty \]

And

**Condition 2.** In a finite time interval, $x(t)$ has a *finite* number of maxima and minima.

**Ex.** An example that violates Condition 2.

\[ x(t) = \sin \left( \frac{2\pi}{T} t \right) \quad 0 < t \leq 1 \]

And

**Condition 3.** In a finite time interval, $x(t)$ has only a *finite* number of discontinuities.

**Ex.** An example that violates Condition 3.

Dirichlet conditions are met for most of the signals encountered in the real world.

Still, convergence has some interesting characteristics:

\[ x_N(t) = \sum_{n=-N}^{N} X_n e^{j \frac{2\pi n}{T} t} \]

As $N \to \infty$, $x_N(t)$ exhibits Gibbs' phenomenon at points of discontinuity. Under the Dirichlet conditions:
- The Fourier series $= x(t)$ at points where $x(t)$ is continuous,
- The Fourier series $= \text{“midpoint”}$ at points of discontinuity.

**Demo:** Fourier series for continuous-time square wave (Gibbs’ phenomenon).

What about Fourier representations of nonperiodic continuous-time signals?

Assuming a finite-energy signal and $T \to \infty$ in the Fourier series, we get $\lim_{T \to \infty} X_n = 0$.

**Trick:** To preserve the Fourier coefficients from disappearing as $T \to \infty$, introduce

$$\tilde{X}_n = TX_n = \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi n}{T}t} dt.$$
Transition to Fourier transform:

\[ X(\Omega) = \lim_{T \to \infty} \tilde{X}_n \]

\[ = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi n}{T}t} dt \]

\[ = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt, \]

where the “discrete” frequency \(2\pi n/T\) becomes the continuous frequency \(\Omega\).

Transition to inverse Fourier transform:

\[ x(t) = \lim_{T \to \infty} \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n}{T}t} = \lim_{T \to \infty} \sum_{n=-\infty}^{\infty} \frac{\tilde{X}_n}{T} e^{j\frac{2\pi n}{T}t} \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t} d\Omega \Leftrightarrow d\Omega = \frac{2\pi}{T}, \Omega = \frac{2\pi n}{T}. \]
Continuous-time Fourier transform (CTFT):

\[
X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt,
\]
\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega.
\]

**Example**: Finite-energy rectangular signal:

\[
X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt
\]
\[
= \int_{-\tau/2}^{\tau/2} Ae^{-j\Omega t}dt
\]
\[
= A\tau \frac{\sin(\Omega\tau/2)}{\Omega\tau/2} \quad \text{real spectrum.}
\]
Remarks:

- In general, Fourier spectrum is complex-valued,

- For real signals, \( X(-\Omega) = X^*(\Omega) \).
Dirac Delta Function

Definition:

\[ \delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1. \]

Do not confuse continuous-time \( \delta(t) \) with discrete-time \( \delta(n) \)!

Sifting property:

\[ \int_{-\infty}^{\infty} f(t) \delta(t - \tau) dt = f(\tau). \]

The spectrum of \( \delta(t - t_0) \):

\[ X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \]
\[ = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\Omega t} dt \]
\[ = e^{-j\Omega t_0}. \]

Delta function in time domain:

\[ \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\Omega t} d\Omega. \]
Delta function in frequency domain:

\[ \delta(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\Omega t} dt = \begin{cases} \infty, & \Omega = 0, \\ 0, & \Omega \neq 0. \end{cases} \]

For signal \( x(t) = Ae^{j\Omega_0 t} \) we have

\[ X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{-j(\Omega-\Omega_0)t} dt = A 2\pi \delta(\Omega - \Omega_0). \]
Harmonic Fourier Pairs

Delta function in frequency domain:

\[ e^{j\Omega_0 t} \leftrightarrow 2\pi \delta(\Omega - \Omega_0), \]
\[ \cos(\Omega_0 t) \leftrightarrow \pi \left[ \delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0) \right], \]
\[ \sin(\Omega_0 t) \leftrightarrow \frac{\pi}{j} \left[ \delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0) \right]. \]
Parseval’s Theorem for CTFT

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \]

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)x^*(\tau)e^{-j\Omega(t-\tau)} dtd\tau \right\} d\Omega \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)x^*(\tau) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\Omega(\tau-t)} d\Omega \right\} dtd\tau \]

\[ = \int_{-\infty}^{\infty} |x(t)|^2 dt. \]
Discrete-time Fourier Transform (DTFT)

Represent continuous signal $x(t)$ via discrete sequence $x(n)$:

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) \delta(t - nT).$$

Substituting this equation into the CTFT formula, we obtain:

$$X(\Omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n) \delta(t - nT) e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t - nT) e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n T}.$$

Switch to the discrete-time frequency, i.e. use $\omega = \Omega T$:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}.$$

$X(\omega)$ is periodic with period $2\pi$:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} e^{-j2\pi n}$$
\[
= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi)n} = X(\omega + 2\pi).
\]

**Trick:** in computing inverse DTFT, use only one period of \(X(\omega)\):

\[
X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{DTFT},
\]

\[
x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad \text{Inverse DTFT}.
\]

**Inverse DTFT:**

**Proof.**

\[
x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x(m) e^{j\omega (n-m)} d\omega
\]

\[
= \sum_{n=-\infty}^{\infty} x(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega (n-m)} d\omega = x(n).
\]

\(\square\)
Fourier Series vs. DTFT

\[ x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n t}, \quad X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi n t} dt \quad \text{FS} \]

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}, \quad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad \text{DTFT} \]

**Observation:** Replacing, in Fourier Series

\[
\begin{align*}
x(t) & \rightarrow X(\omega), \\
X_n & \rightarrow x(n), \\
t & \rightarrow -\omega, \\
T & \rightarrow 2\pi,
\end{align*}
\]

we obtain DTFT!

**An important conclusion:** DTFT is equivalent to Fourier series but applied to the “opposite” domain. In Fourier series, a periodic continuous signal is represented as a sum of exponentials weighted by discrete Fourier (spectral) coefficients. In DTFT, a periodic continuous spectrum is represented as a sum of exponentials, weighted by discrete signal values.
Remarks:

- DTFT can be derived directly from the Fourier series,
- All Fourier series results can be applied to DTFT
- Duality between time and frequency domains.
Parseval’s Theorem for DTFT

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n)x^*(m)e^{-j\omega(n-m)} d\omega$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n)x^*(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega(n-m)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(n)x^*(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega(n-m)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} |x(n)|^2.$$
When Does DTFT Exist (i.e. $|X(\omega)| < \infty$)?

**Sufficient condition:**

$$
\sum_{n=-\infty}^{\infty} |x(n)| < \infty.
$$

$$
|X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n)| \left| e^{-j\omega n} \right| 1
$$

$$
= \sum_{n=-\infty}^{\infty} |x(n)| < \infty.
$$

**Example:** Finite-energy rectangular signal:

$$
|X(\omega)| = \sum_{n=-N/2}^{N/2} Ae^{-j\omega n} = A \sum_{n=-N/2}^{N/2} e^{-j\omega n}
$$

$$
= A(N+1) \frac{\sin\left(\frac{N+1}{2}\omega\right)}{(N+1)\sin\left(\frac{\omega}{2}\right)}
$$
\[ A(N + 1) \left( \frac{\sin\left(\frac{N+1}{2}\omega\right)}{(N + 1)^{\frac{3\omega}{2}}} \right) \text{ for } \omega \ll \pi \]

Both functions look very similar in their “mainlobe” domain.
DTFT — Convolution Theorem

If \( X(\omega) = \mathcal{F}\{x(n)\} \) and \( H(\omega) = \mathcal{F}\{h(n)\} \) and

\[
y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \{x(n)\} \ast \{h(n)\}
\]

then \( Y(\omega) = \mathcal{F}\{y(n)\} = X(\omega)H(\omega) \).

\[
Y(\omega) = \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x(k)h(n-k) \right\} e^{-j\omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x(k)h(m)e^{-j\omega(m+k)} \right\}
\]

\[
= \left\{ \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \right\} \left\{ \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m} \right\}
\]

\[
= X(\omega)H(\omega).
\]

Windowing theorem: If \( X(\omega) = \mathcal{F}\{x(n)\}, \ W(\omega) = \mathcal{F}\{w(n)\}, \) and \( y(n) = x(n)w(n), \) then

\[
Y(\omega) = \mathcal{F}\{y(n)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda) d\lambda.
\]
Frequency-Domain Characteristics of LTI Systems

Recall impulse response $h(n)$ of an LTI system:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n - k).$$

Consider input sequence $x(n) = e^{j\omega n}, -\infty < n < \infty$.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = e^{j\omega n}H(\omega).$$

The complex function

$$\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

is called the frequency response or the transfer function of the system.

- Impulse response and transfer function represent a DTFT pair $\Longrightarrow H(\omega)$ is a periodic function.
• Transfer function shows how different input frequency components are changed (e.g. attenuated) at system output.

• $Y(\omega) = X(\omega)H(\omega)$ implies that an LTI system cannot generate any new frequencies, i.e. it can only amplify or reduce/remove frequency components of the input. Conversely, if a system generates new frequencies, then it is not LTI!

• Systems that are not LTI do not have a meaningful frequency response.
Elements of Sampling Theory

Preliminaries

How are CTFT and Fourier series related for periodic signals? Consider a continuous-time signal $x_c(t)$ with CTFT

$$X(\Omega) = 2\pi \delta(\Omega - \Omega_0).$$

Then

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\Omega - \Omega_0)e^{j\Omega t} d\Omega = e^{j\Omega_0 t}.$$  

We know: periodic signal has line equispaced spectrum. Let $X(\Omega)$ be a linear combination of impulses equally spaced in frequency:

$$X(\Omega) = \sum_{n=\infty}^{\infty} 2\pi X_n \delta(\Omega - n\Omega_0).$$  \hspace{1cm} (1)

Using inverse CTFT, i.e. applying it to each term in the sum, we obtain:

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi X_n \delta(\Omega - n\Omega_0)e^{j\Omega t} d\Omega$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{j n\Omega_0 t} \quad \text{Fourier series!}$$
CTFT of a periodic signal with Fourier-series coefficients \( \{X_n\} \) can be interpreted as a train of impulses occurring at the harmonically-related frequencies with the weights \( \{2\pi X_n\} \).

How about the following signal (periodic impulse train), defined as

\[
s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \?
\]

**Note:** The above periodic impulse train does not satisfy the Dirichlet conditions. Hence, its CTFT is introduced and understood in a limiting sense.

Here are couple of useful expressions for the periodic impulse train:

\[
\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j n (2\pi/T) t},
\]

\[
\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi n}{T}\right) = \sum_{k=-\infty}^{\infty} e^{-j \Omega k T}.
\]

Also, the Fourier transform of a periodic impulse train is a periodic impulse train:

\[
\sum_{k=-\infty}^{\infty} \delta(t - kT) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi n}{T}\right).
\]
Proof.

The impulse train \( s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \) is a periodic signal with period \( T \) \( \implies \) we can apply Fourier series and find the Fourier coefficients:

\[
S_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\frac{2\pi n}{T}t} dt = \frac{1}{T}.
\]

Hence

\[
s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn(2\pi/T)t}.
\]

Also

\[
S(\Omega) = \int_{-\infty}^{\infty} s(t) e^{-j\Omega t} dt = \sum_{k=-\infty}^{\infty} e^{-j\Omega kT}.
\]
From (1), we obtain

\[ S(\Omega) = \sum_{n=-\infty}^{\infty} 2\pi S_n \delta(\omega - n \frac{\Omega_0}{2\pi T}) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega - \frac{2\pi n}{T}). \]
The Sampling Theorem

Introduce the “modulated” signal

\[ x_s(t) = x_c(t)s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT). \]

Since \( x_c(t) \delta(t - t_0) = x_c(t_0) \delta(t - t_0) \), we obtain

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(n) \delta(t - nT). \]

Using this “modulated” signal, we describe the sampling operation.
It turns out that the problem is much easier to understand in the frequency domain. Hence, we compute the Fourier transform of \( x_s(t) \). Looking at each term of the summation,
we have from the frequency-shift theorem:

\[ x_c(t) e^{jn(2\pi/T)t} \leftrightarrow X_c\left(\Omega - n\frac{2\pi}{T}\right). \]

Hence, the Fourier transform of the sum is

\[ X_s(\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(\Omega - \frac{2\pi n}{T}\right). \]

Recall that:

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(n) \delta(t - nT). \]

Taking the FT of the above expression, we obtain another expression for \( X_s(\Omega) \):

\[ X_s(\Omega) = \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t - nT)e^{-j\Omega t} dt \]

\[ = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega NT} = \underbrace{X(\omega)}_{\text{DTFT}\{x(n)\}} \bigg|_{\omega = \Omega T}. \]

By sampling, we throw out a lot of information: all values of \( x(t) \) between the sampling points are lost.
**Question:** Under which conditions can we reconstruct the original continuous-time signal $x(t)$ from the sampled signal $x_s(t)$?

**Theorem.** Suppose $x(t)$ is bandlimited, so that $X(\Omega) = 0$ for $|\Omega| > \Omega_M$. Then $x(t)$ is uniquely determined by its samples $\{x(nT)\} = \{x(n)\}$ if

$$\Omega_s \equiv \frac{2\pi}{T} > 2\Omega_M \equiv \text{the Nyquist rate.}$$
Frequency-Domain Effect for Nyquist Sampling

\[(2\pi/T = 2\Omega_M)\]
Frequency-domain Effect for Sampling Faster than Nyquist \((2\pi/T > 2\Omega_M)\)
Frequency-domain Effect for Sampling Slower than Nyquist \( (2\pi/T < 2\Omega_M) \)
Introduce a lowpass filtering operation. The spectrum of the filtered signal:

\[ X_f(\Omega) = H_{LP}(\Omega)X_s(\Omega) \]

where \( H_{LP}(\Omega) \equiv \) ideal lowpass filter:

\[
H_{LP}(\Omega) = \begin{cases} 
T, & -\Omega_c \leq \Omega \leq \Omega_c, \\
0, & \text{otherwise}
\end{cases}
\]

with the cut-off frequency \( \Omega_c \). How to reconstruct a bandlimited signal from its samples in the time domain?

Having a signal sampled at a rate higher than the Nyquist rate and infinite number of its discrete values, the signal can be exactly recovered as

\[
x_f(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}
\]

ideal interpolation formula.

**Proof.** Start from: \( X_f(\Omega) = X_s(\Omega)H_{LP}(\Omega) \). In time domain

\[
x_f(t) = \{x_s(t)\} \ast \{h_{LP}(t)\}
\]

\[
= \left\{ \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \right\} \ast \{h_{LP}(t)\}
\]
\[
= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_c(nT)\delta(\tau - nT)h_{LP}(t - \tau)\,d\tau
\]

\[
= \sum_{n=-\infty}^{\infty} x(n)h_{LP}(t - nT).
\]

Ideal transfer function:

\[
H_{LP}(\Omega) = \begin{cases} 
T, & -\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T}, \\
0, & \text{otherwise}
\end{cases}
\]

Ideal impulse response:

\[
h_{LP}(t) = \frac{\sin(\pi t/T)}{\pi t/T}.
\]

Now, insert \(h_{LP}(t)\) into the equation for \(x_f(t)\).  \(\Box\)
Representations of Narrowband Signals

Narrowband signals have small bandwidth compared to the band center (carrier) frequency.

\[ B \ll F_0. \]

\[ x_c(t) = a(t) \cos[2\pi F_0 t + \theta(t)]. \]

- **Amplitude modulation**
- **Phase modulation**
The above representation can be used to describe any signal, but it makes sense only if $a(t)$ and $\theta(t)$ vary slowly compared with $\cos(2\pi F_0 t)$, or, equivalently, $B \ll F_0$.

- Complex-envelope and
- Quadrature-component

representations of narrowband signals.

**Complex-envelope representation:**

\[
x_c(t) = \text{Re}\{a(t) \exp(j[\Omega_0 t + \theta(t)])}\}
= \text{Re}\{a(t) \exp[j\theta(t)] \exp(j\Omega_0 t)\}
= \tilde{x}_c(t).
\]

The complex-valued signal $\tilde{x}_c(t)$ contains both the amplitude and phase variations of $x_c(t)$, and is hence referred to as the **complex envelope** of the narrowband signal $x_c(t)$. 
Quadrature-component representation:

\[ x_c(t) = x_{cI}(t) \cos(\Omega_0 t) - x_{cQ}(t) \sin(\Omega_0 t). \]

\( x_{cI}(t) \) and \( x_{cQ}(t) \) are termed the in-phase and quadrature components of narrowband signal \( x_c(t) \), respectively.

Note that

\[ \tilde{x}_c(t) = x_{cI}(t) + jx_{cQ}(t). \]

If we “blindly” apply the Nyquist theorem, we would choose

\[ F_N = 2(F_0 + \frac{B}{2}) \approx 2F_0 \quad \text{for} \quad B \ll F_0. \]

However, since the effective bandwidth of \( x_c(t) \) [and \( \tilde{x}_c(t) \)] is \( B/2 \), the optimal rate should be \( B! \).

Recall

\[ x_c(t) = a(t) \cos[2\pi F_0 t + \theta(t)] \]

\[ = a(t) \cdot \exp\{j[\Omega_0 t + \theta(t)]\} + \exp\{-j[\Omega_0 t + \theta(t)]\} \]

\[ = \frac{a(t) \exp[j\theta(t)]}{2} \exp(j\Omega_0 t) + \frac{a(t) \exp[-j\theta(t)]}{2} \exp(-j\Omega_0 t) \]

\[ = \frac{1}{2} \tilde{x}_c(t) \exp{j\Omega_0 t} + \frac{1}{2} \tilde{x}_c^*(t) \exp(-j\Omega_0 t) \]
and hence

$$X_c(\Omega) = \frac{1}{2}[\tilde{X}_c(\Omega - \Omega_0) + \tilde{X}_c^*(-\Omega - \Omega_0)],$$

implying that $\tilde{x}_c$ is a baseband complex-valued signal (occupying the band $[-B/2, B/2]$):

$$\tilde{x}_c(t) = \sum_{n=-\infty}^{\infty} \tilde{x}_c(\frac{n}{B}) \frac{\sin[\pi B(t - n/B)]}{\pi B(t - n/B)}$$
Now,

\[ x_c(t) = \text{Re}\{\tilde{x}_c(t) \exp(j2\pi F_0 t)\} \]

\[ = \text{Re}\left\{ \sum_{n=-\infty}^{\infty} \tilde{x}_c\left(\frac{n}{B}\right) \frac{\sin[\pi B(t - n/B)]}{\pi B(t - n/B)} \exp(j2\pi F_0 t)\right\} \]

\[ = \text{Re}\left\{ \sum_{n=-\infty}^{\infty} a\left(\frac{n}{B}\right) \exp\{j[\theta(n/B) + 2\pi F_0 t]\} \frac{\sin[\pi B(t - n/B)]}{\pi B(t - n/B)} \right\} \]

\[ = \sum_{n=-\infty}^{\infty} a\left(\frac{n}{B}\right) \cos[2\pi F_0 t + \theta(n/B)] \frac{\sin[\pi B(t - n/B)]}{\pi B(t - n/B)} \]

\[ = \sum_{n=-\infty}^{\infty} \left[ x_{ci}\left(\frac{n}{B}\right) \cos(2\pi F_0 t) - x_{cQ}\left(\frac{n}{B}\right) \sin(2\pi F_0 t) \right] \]

\[ \cdot \frac{\sin[\pi B(t - n/B)]}{\pi B(t - n/B)} . \]
\[ X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}. \]

Relationship between the \(z\) transform and DTFT: substitute \(z = re^{j\omega}\),

\[
X(z)\big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \{x(n)r^{-n}\}e^{-j\omega n} = \mathcal{F}\{x(n)r^{-n}\} \implies
\]

The \(z\) transform of an arbitrary sequence \(x(n)\) is equivalent to DTFT of the exponentially weighted sequence \(x(n)r^{-n}\).

If \(r = 1\) then

\[
X(z)\big|_{z=e^{j\omega}} = X(\omega) = \mathcal{F}\{x(n)\} \implies
\]

DTFT corresponds to \(z\) transform with \(|z| = 1\).

**Notation:** Observe that \(X(e^{j\omega}) \equiv X(\omega)\).
The $z$ transform reduces to the DTFT for values of $z$ on the unit circle:

**Question:** When does the $z$ transform converge?

Region of convergence (ROC) ≡ range of values of $z$ for which $|X(z)| < \infty$.

**Example:** The $z$ transform of the signal $x(n) = a^n u(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$
For convergence, we require that

\[ \sum_{n=0}^{\infty} |a z^{-1}|^n < \infty, \]

which holds if \(|a z^{-1}| < 1\) or, equivalently, \(|z| > |a|\). Note:

\[ X(z) = \frac{1}{1 - a z^{-1}}. \]

**Example:** The \(z\) transform of the signal

\[ x(n) = -a^n u(-n - 1) = \begin{cases} 0, & n \geq 0, \\ -a^n, & n \leq -1 \end{cases} \]

is

\[
X(z) = - \sum_{n=-\infty}^{-1} a^n z^{-n} = - \sum_{n=1}^{\infty} a^{-n} z^n = - \sum_{n=1}^{\infty} (a^{-1} z)^n
\]

\[ = \frac{-a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} \equiv \text{same as previous ex.} \]

But, ROC is now \(|z| < |a|\).

**Remark:** A discrete-time signal \(x(n)\) is uniquely determined by its \(z\) transform \(X(z)\) and its ROC.
ROC Properties

- The ROC of $X(z)$ consists of a ring in the $z$ plane centered about the origin,

- The ROC does not contain any poles,

- If $x(n)$ is of finite duration, then the ROC is the entire $z$ plane, except possibly $z = 0$ and/or $z = \infty$,

- If $x(n)$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of $z$ for which $|z| > r_0$ will also be in the ROC (need to check $z = \infty$),

- If $x(n)$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of $z$ for which $0 < |z| < r_0$ will also be in the ROC (need to check $z = 0$),

- If $x(n)$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will be a ring in the $z$ plane that includes the circle $|z| = r_0$ (we can represent this sequence as right-sided sequence + left-sided sequence).
**Z Transform (cont.)**

**Inverse Z transform:**

Recall that

\[ X(z) \bigg|_{z=re^{j\omega}} = \mathcal{F}\{x[n]r^{-n}\}. \]

Applying the inverse DTFT, we get

\[
x[n] = r^n \mathcal{F}^{-1}\{X(r e^{j\omega})\}
= r^n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} X(r e^{j\omega}) e^{j\omega n} d\omega
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(z) z^{-n} d\omega
= \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz \quad \leftarrow \, dz = jre^{j\omega} d\omega.
\]

**Comments:**

- \(\oint \cdots \, dz\) denotes integration around a closed circular contour centered at the origin and having radius \(r\),
- \(r\) must be chosen so that the contour of integration \(|z| = r\) belongs to the ROC,
- contour integration in complex plane may be a complicated task; simpler alternative procedures exist for obtaining a sequence from a Z transform.
LTI system analysis:

\[ y(n) = \{h(n)\} \star \{x(n)\} \leftrightarrow Y(z) = H(z)X(z) \]

Results:

- A discrete-time LTI system is causal if and only if the ROC of its transfer function is the exterior of a circle including infinity.

- A discrete-time LTI system is stable if and only if the ROC of its transfer function includes the unit circle \( |z| = 1 \).
Rational $Z$ transforms

Recall LCCD equations of ARMA processes

\[ \sum_{k=0}^{N} a_k y(n - k) = \sum_{k=0}^{M} b_k x(n - k). \]

Taking $z$-transforms of both sides, we get

\[ \sum_{k=0}^{N} Z\{a_k y(n - k)\} = \sum_{k=0}^{M} b_k Z\{x(n - k)\}, \]

yielding

\[ Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}. \]

Hence, the transfer function of an ARMA process is

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \]