

## Real-Time Electricity Markets - 2

### 1.0 Social surplus formulation

In the formulation of the previous notes, we minimized cost. In doing so, we are maximizing social surplus under the condition that demand is price insensitive. We can see this by recall the problem being solved by the benevolent dictator is to maximize the sum of consumer utility and supplier profit, i.e.,

$$\max_P \{U(P) + pP - C(P)\} \quad (1)$$

or, since  $U(P) = v(P) + m - pP$ , we see the problem is

$$\max_P \{v(P) + m - pP + pP - C(P)\} \quad (2)$$

Eliminating the  $pP$  terms & removing the constant  $m$ :

$$\max_P \{v(P) - C(P)\} \quad (3)$$

Under the condition that demand is price-insensitive, then the demand function  $v'(P)$  is an impulse, indicating that the consumer will pay anything to obtain the amount  $P$ . This also means that the value to them of obtaining more than  $P$  is the same as the value of obtaining  $P$ . Therefore,  $v(P)$  is constant. This implies that the above maximization problem is just

$$\max_P \{-C(P)\} \quad (4a)$$

Or

$$\min_P \{C(P)\} \quad (4b)$$

which is what we solved in the last set of notes.

Actual markets today have provision for load-serving entities (LSEs) to bid into the market, just as resources can make offers. Resource offers and demand bids are illustrated in Figs. 1a and 1b [1]. Resource offers may be “block” (as shown) or “slope.” These offers and bids correspond to the what we have called  $C'(P)$  and  $v'(P)$ , respectively.

Segment	Offer MW to \$/MWh	
	MW	\$/MWh
1	75	\$24.50
2	85	\$24.75
3	95	\$25.00
4	105	\$25.50
5	125	\$25.75
6	135	\$26.00

**BLOCK**

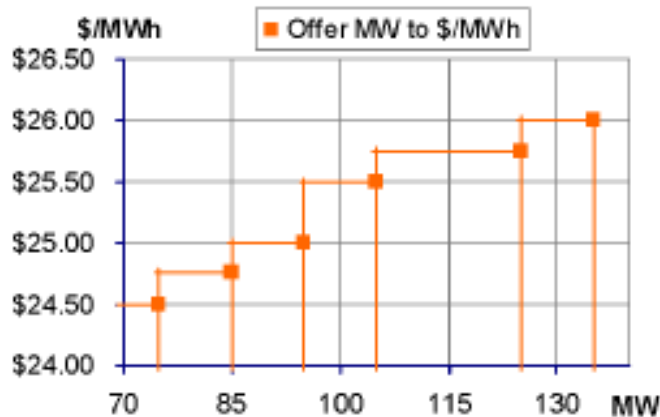


Fig. 1a: Resource offer

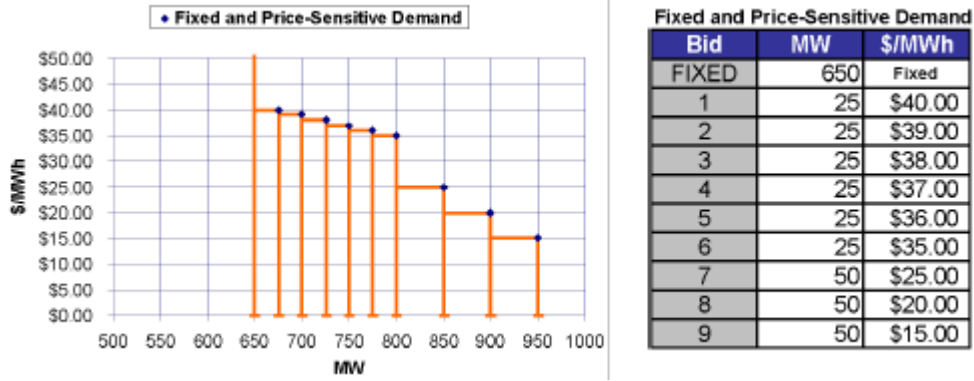


Fig. 1b: Demand bid

So we want to account for the possibility that the consumer will desire to adjust their demand as a function of the price. In this case, we must include the consumers' utilities in the objective function, leading us back to the objective function used in the formulation for our present problem, which is:

$$\max_P \{v(P) - C(P)\} \quad (5a)$$

In the case where we have multiple consumers and suppliers, then (5a) becomes

$$\max_P \left\{ \sum_{k \in \{LoadBuses\}} v_k(P_k) - \sum_{k \in \{GenBuses\}} C_k(P_k) \right\} \quad (5b)$$

## 2.0 LPOPF with Consumer Utility

We saw in previous notes that the efficient market occurs when we maximize social surplus, which we expressed as  $v(P)-C(P)$ , where

- $v(P)$  quantifies, in dollars, the satisfaction associated with the amount of energy consumed,  $P$ , and is the utility function for energy;
- $C(P)$  is the cost of producing the energy  $P$ .

We assume the network has  $n$  buses and  $m$  branches, and that there can be both generation and demand at each bus.

To simplify, we use a piecewise linear approximation of the cost and utility curves with only 1 “piece” per curve. Thus, each generation unit and each consumer is represented in the objective function by a constant times the MW output for that unit or the MW consumption by that consumer. So here is the formal statement of our problem:

$$\min \sum_{k \in \{generator\_buses\}} s_{gk} P_{gk} + \sum_{k \in \{load\_buses\}} -s_{dk} P_{dk} \quad (6)^1$$

Subject to:

$$\underline{P} = \underline{B}' \underline{\theta} \quad (7)^2$$

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \quad (8)^3$$

$$- \underline{P}_{B,max} \leq \underline{P}_B \leq \underline{P}_{B,max} \quad (9)^4$$

$$0 \leq P_{gk} \leq P_{gk,max}, \forall k \in \{generator\_buses\} \quad (10)^5$$

$$0 \leq P_{dk} \leq P_{dk,max}, \forall k \in \{load\_buses\} \quad (11)^6$$

where

$$P_k = P_{gk} - P_{dk}, k = 1, \dots, N \quad (12)^7$$

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<sup>1</sup> We want to maximize social surplus as defined by  $\sum U_k - \sum C_k$ , but this is the same as minimizing  $\sum C_k - \sum U_k$ . We make this change because the LP available to us in Matlab is a minimizing LP.

<sup>2</sup> So these are the DC power flow equations to represent the network. However, we must include all nodal injections  $P_1, \dots, P_N$  and all angles  $\theta_1 \dots \theta_N$  in this set of equations.

<sup>3</sup> These are the equation to get the line flows. Again, we need to include all angles  $\theta_1 \dots \theta_N$  in the vector  $\underline{\theta}$ .  $\underline{D}$  is an  $m \times m$  matrix of all zeros except the diagonal, where the  $m^{\text{th}}$  element is the negated susceptance of branch  $m$ .

$\underline{A}$  is the  $m \times n$  incidence matrix for the network.

<sup>4</sup> These are the limits on the line flows. Notice that there is only one set of circuit ratings  $\underline{P}_{B,max}$ , but they must be enforced as a limit if the flow is in one direction or in the other.

<sup>5</sup> These are the limits on the linear cost curve variables.

<sup>6</sup> These are the limits on the linear utility curve variables.

<sup>7</sup> This equation relates the variables used in the cost curves ( $P_{kj}$ ) to the variables used in the DC power flow equations ( $P_k$ ).

We identify the decision and coefficient vectors as:

$$\underline{x} = \begin{bmatrix} P_{g1} \\ \vdots \\ P_{gn} \\ P_{d1} \\ \vdots \\ P_{dn} \\ P_{b1} \\ \vdots \\ P_{bm} \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}; \quad \underline{c} = \begin{bmatrix} S_{g1} \\ \vdots \\ S_{gn} \\ -S_{d1} \\ \vdots \\ -S_{dn} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad (13)$$

We are now in a position to state the LPOPF more compactly.

$$\text{Max } \underline{c}^T \underline{x} \quad \text{Subject to:} \quad (14)$$

$$\underline{A}_{eq} \underline{x} = \underline{b}_{eq}, \quad \underline{x}_{\min} \leq \underline{x} \leq \underline{x}_{\max} \quad (15)$$

where the equality constraints in the  $\underline{A}_{eq}$  matrix equation model the line flow equations and DC power flow equations.

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \quad (16)$$

$$-\underline{P} + \underline{B}' \underline{\theta} = \underline{0} \quad (17)$$

and the inequality constraints are given by:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ -P_{b1,ma} \\ \vdots \\ -P_{bm,max} \\ -\pi \\ \vdots \\ -\pi \end{bmatrix} \leq \begin{bmatrix} P_{g1} \\ \vdots \\ P_{gn} \\ P_{d1} \\ \vdots \\ P_{dn} \\ P_{b1} \\ \vdots \\ P_{bm} \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} \leq \begin{bmatrix} P_{g1,max} \\ \vdots \\ P_{gn,max} \\ P_{d1,max} \\ \vdots \\ P_{dn,max} \\ P_{b1,max} \\ \vdots \\ P_{bm,max} \\ \pi \\ \vdots \\ \pi \end{bmatrix} \quad (18)$$

Some particular notes about the above problem statement:

- The upper right-hand  $m \times n$  submatrix of  $\underline{A}_{eq}$  is  $\underline{D} \times \underline{A}$ .
- The lower right-hand  $n \times n$  submatrix of  $\underline{A}_{eq}$  is  $\underline{B}'$ .
- The right-hand-side of the equality constraint equation,  $\underline{b}_{eq}$ , is all zeroes because we now have variables for the demand which means it must be included in the  $\underline{A}_{eq}$  matrix instead of being a fixed constant (and therefore represented in the  $\underline{b}_{eq}$  vector).

### 3.0 Example: Unconstrained transmission

We illustrate using a 4 bus example. The one-line diagram for the example system is given in Fig. 2. Directionality on the branches is denoted by arrows and will be used to form the branch flow equations, where the  $k^{th}$  branch flow is denoted by  $P_{bk}$ .

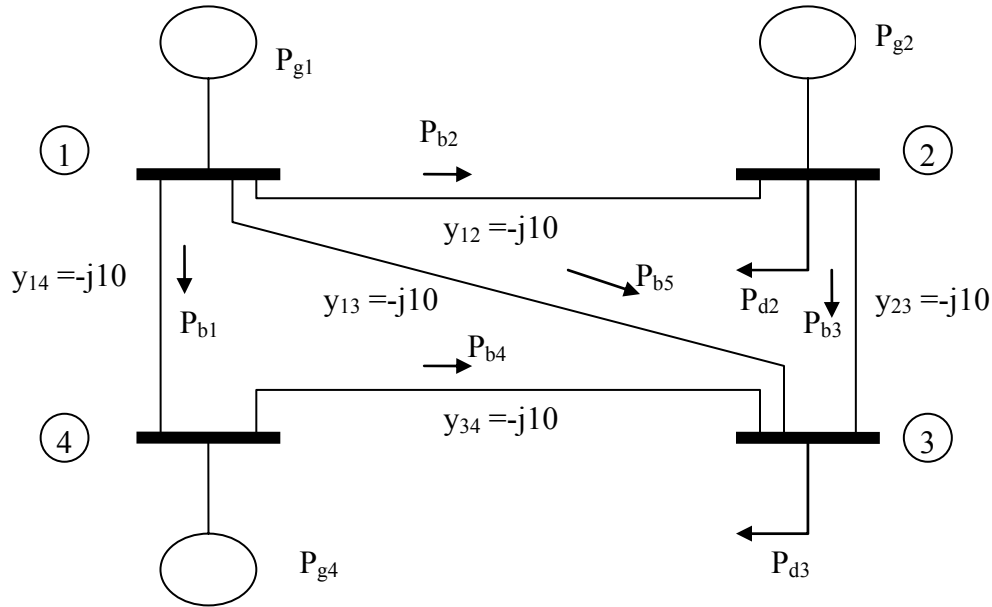


Fig. 2: One line diagram for example system

The cost-curves for the three generators are given by

$$C_1(P_{g1}) = s_{g1}P_{g1}$$

$$C_2(P_{g2}) = s_{g2}P_{g2}$$

$$C_4(P_{g4}) = s_{g4}P_{g4}$$

where the generation variables are in pu (100 MVA base) and the coefficients are

$$s_{g1}=1307 \text{ \$/pu-hr}$$

$$s_{g2}=1211 \text{ \$/pu-hr}$$

$$s_{g4}=1254 \text{ \$/pu-hr}$$

The generation limits in per-unit are

$$0.50 \leq P_{g1} \leq 2.00$$

$$0.375 \leq P_{g2} \leq 1.50$$

$$0.45 \leq P_{g4} \leq 1.80$$

The linearized functions for demand bids are:

$$v_2(P_{d2}) = s_{d2}P_{d2}$$

$$v_3(P_{d3}) = s_{d3}P_{d3}$$

where the load variables are in per-unit and the coefficients are

$$s_{d2}=-1300 \text{ \$/pu-hr}$$

$$s_{d3}=-1200 \text{ \$/pu-hr}$$

The constraints are

$$1.00 \leq P_{d2} \leq 2.00$$

$$2.00 \leq P_{d3} \leq 3.00$$

**Objective function:** Let's explicitly write out the solution vector.

$$\underline{x} = \begin{bmatrix} \underline{P}_g \\ \underline{P}_d \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

So, using these coefficients, the objective function is:

$$\underline{Z}(\underline{x}) = \underline{c}^T \underline{x} = [1307 \quad 1211 \quad 1254 \quad -1300 \quad -1200 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

**Equality constraints:** The equality constraints are given in eqs. (16) and (17), repeated here for convenience:

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \quad (16)$$

$$-\underline{P} + \underline{B}' \underline{\theta} = \underline{0} \quad (17)$$

We need to build these equality constraints into a matrix form of  $\underline{A}_{eq} \underline{x} = \underline{b}_{eq}$ . We begin by noting dimensions.

- Columns: Since the solution vector  $\underline{x}$  is 14x1,  $\underline{A}_{eq}$  must have 14 columns in order to pre-multiply  $\underline{x}$ .
- Rows: Since there are 5 branches, eq. (16) will contribute 5 rows to  $\underline{A}_{eq}$ . Since there are 4 buses, eq. (17) will contribute 4 rows to  $\underline{A}_{eq}$ . So  $\underline{A}_{eq}$  will have total of 9 rows.

Therefore, the dimensions of  $\underline{A}_{eq}$  will be  $9 \times 14$ .

We begin with the line flow equations, eq. (16). The  $\underline{D}$  matrix is diagonal with element  $(m,m)$  containing the negated susceptance of branch  $m$ , which is:

$$\underline{D} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

The  $m \times n$  node-arc incidence matrix  $\underline{A}$  is constructed from the network of Fig. 2 by identifying row  $k$  with branch  $k$ , where branch  $k$  is terminated by buses  $i$  and  $j$ , with bus  $i$  being the sending-end bus (according to directionality given in Fig. 2) and placing a “1” in the element  $(k,i)$  and a “-1” in element  $(k,j)$ .

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The  $\underline{D} \times \underline{A}$  product required by eq. (16) is then given by:

$$\underline{D} \times \underline{A} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & -I \\ I & -I & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & -10 \\ 10 & -10 & 0 & 0 \\ 0 & 10 & -10 & 0 \\ 0 & 0 & -10 & 10 \\ 10 & 0 & -10 & 0 \end{bmatrix}$$

So based on eq. (16) and the solution vector, we can see that these elements will occupy the upper right hand corner of  $\underline{A}_{eq}$ . So that will take care of the last 4 columns in the first 5 rows.

But what about the first 10 columns? These are the elements in the line flow equations that multiply the variables  $P_{g1}, P_{g2}, P_{g4}, P_{d2}, P_{d3}, P_{B1}, P_{B2}, P_{B3}, P_{B4}, P_{B5}$ .

Since we do not use the generation or demand variables within the line flow equations, the first 5 columns of these top 5 rows will be zeros. The last 5 columns in these top 5 rows will also be zeros, except the one element in each of these rows that multiply the corresponding line flow variable, and that element will be -1.

Finally, with respect to these top 5 equations, eq. (16) indicates that the right-hand-side will be 0 for each of them.

Thus, we can now write down all elements in the first 5 rows of our matrix, as follows:

$$\underline{A}_{eq} \underline{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we need to write the last 4 equations. These are the DC power flow equations corresponding to eq. (17).

The solution vector contains all 4 angles, and therefore the DC power flow matrix  $\underline{B}'$  needs to be  $4 \times 4$ . This DC power flow matrix  $\underline{B}'$  is given below:

$$\underline{B}' = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 20 & -10 & 0 \\ -10 & -10 & 30 & -10 \\ -10 & 0 & -10 & 20 \end{bmatrix}$$

So based on eq. (17) and the solution vector, we can see that this matrix will occupy the lower right hand side of the  $\underline{A}_{eq}$  matrix. So that will take care of the last 4 columns in the bottom 4 rows. The resulting matrix appears as:

$$\underline{A}_{eq} \underline{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ - & - & - & - & - & - & - & - & - & - & 30 & -10 & -10 & -10 \\ - & - & - & - & - & - & - & - & - & - & -10 & 20 & -10 & 0 \\ - & - & - & - & - & - & - & - & - & - & -10 & -10 & 30 & -10 \\ - & - & - & - & - & - & - & - & - & - & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Once again, we need to consider the first eight columns. Columns 6-10 correspond to the line flow variables, which do not appear in the DC power flow equations, so these will be zero.

$$\underline{A}_{eq} \underline{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\ - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\ - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first three columns multiply the generation variables  $P_{g1}$ ,  $P_{g2}$ , and  $P_{g4}$ , and columns 4 and 5 multiply the load variables  $P_{d2}$  and  $P_{d3}$ .

However, the DC power flow equations, eq. (17), require the *negative* of the *injections* for all buses, and the injections are the generation minus the load, i.e.,  $P_{gk} - P_{dk}$ . So we want to model  $-P_{gk} + P_{dk}$  on the left-hand-side in the last 4 rows. This is done by placing a -1 and +1 in the appropriate location.

$$\underline{A}_{eq} \underline{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Inequality constraints:** The inequality constraints are simple, as given below. Notice that the -500 to 500 constraints on line flows imply we are modeling no transmission constraints.

$$\begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ 1 \\ 2 \\ -500 \\ -500 \\ -500 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \leq \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 2 \\ 3 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

**Solution by Matlab:** The code for solving this linear program using Matlab is given below:

```

%Build objective function vector.
c=[1307 1211 1254 -1300 -1200 0 0 0 0 0 0 0 0 0 0]';

%Build A matrix for inequality constraints Ax<b.
A=[];
%Build b, the right-hand-side of inequality
constraints.
b=[];

%Build Aeq matrix for equality constraints.
Aeq=[0 0 0 0 0 -1 0 0 0 0 10 0 0 -10;
      0 0 0 0 0 0 -1 0 0 0 10 -10 0 0;
      0 0 0 0 0 0 0 -1 0 0 0 10 -10 0;
      0 0 0 0 0 0 0 0 -1 0 0 0 -10 10;
      0 0 0 0 0 0 0 0 0 -1 10 0 -10 0;
      -1 0 0 0 0 0 0 0 0 0 30 -10 -10 -10;
      0 -1 0 1 0 0 0 0 0 0 -10 20 -10 0;
      0 0 0 0 1 0 0 0 0 0 -10 -10 30 -10;
      0 0 -1 0 0 0 0 0 0 0 -10 0 -10 20];

%Build right-hand side of equality constraint. It will
be vector of zeros
beq=zeros(9,1);

%Build upper and lower bounds on decision variables.
LB=[.50 .375 .45 1 2 -500 -500 -500 -500 -500 -
pi -pi -pi -pi]';
UB=[2.00 1.50 1.80 2 3 500 500 500 500 500
pi pi pi pi]';
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=linprog(c,A,b,Aeq,beq,L
B,UB);
% Compute settlements assuming pay-at-bid/offer:
dollars=c.*X;

```

The solution vector  $\underline{x}$  is given below. The limits on the variables are also repeated here so that it is easy to see which ones are at their limit.

$$\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 1.8 \\ 1.8 \\ 2.0 \\ -0.5875 \\ 0.4625 \\ 0.1625 \\ 1.2125 \\ 0.6250 \\ 0.0125 \\ -0.0338 \\ -0.05 \\ 0.0712 \end{bmatrix} \leq \begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ 1 \\ 2 \\ -500 \\ -500 \\ -500 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \leq \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 2 \\ 3 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

The solution is provided in Fig. 3.

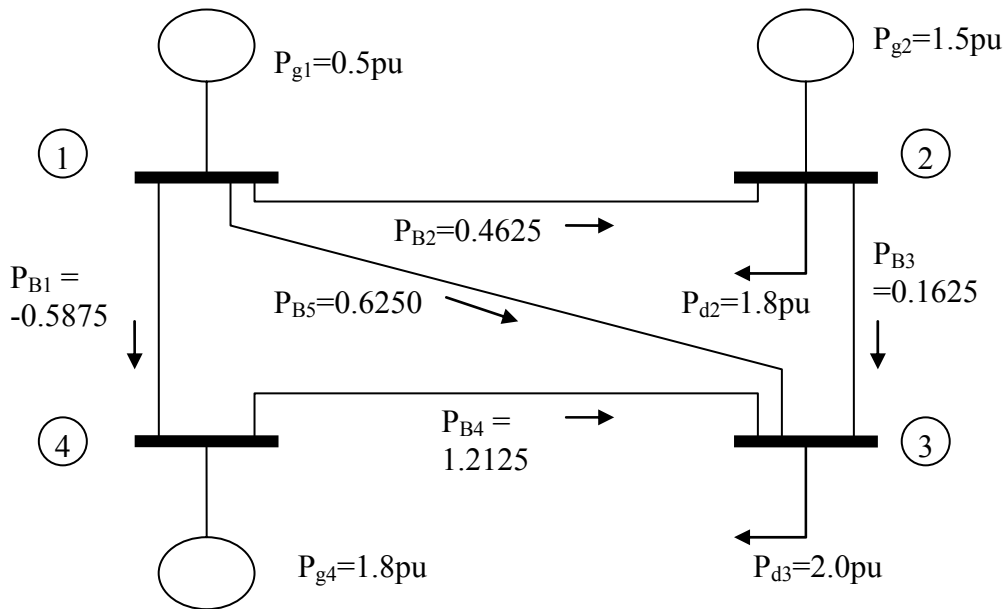


Fig. 3

One can easily check to see that the power is conserved at the buses.

**Objective function value:** The objective function that Matlab provides (FVAL) is  $Z=-12.8$  \$/hr. This is negative of the social surplus (Matlab requires all problems to be minimization problems, so we had to minimize the negative of the social surplus in order to maximize social surplus).

So the social surplus (Total Utility of Load less Total Cost of Supply) is \$12.80. Not too much! This is because the consumers are valuing the energy at just a little above cost.

- If we changed the utility function coefficients to -1500 and -1400, from -1300 and -1200, respectively, the social surplus would change to \$904/hr.
- If we changed utility function coefficients to -1000 and -900, respectively, the social surplus would be -\$924/hr, indicating the cost of supply is more than the utility of consumption, and the only reason any power is being consumed is the lower bound constraints we have placed on generation and demand.

**Lagrange multipliers:** Now let's investigate Lagrange multipliers for this case, assuming infinite capacity lines. These Lagrange multipliers (the same as the dual variables), are given in Table 2.

Table 2: Lagrange multipliers for  $\infty$  transm. capacity

Equality constraints		Lower bounds		Upper bounds	
Equation	Value*10 <sup>3</sup>	Variable	value	variable	value
P <sub>B1</sub>	-0.0000	P <sub>g1</sub>	7.0000	P <sub>g1</sub>	0.0000
P <sub>B2</sub>	-0.0000	P <sub>g2</sub>	0.0000	P <sub>g2</sub>	89.0000
P <sub>B3</sub>	0.0000	P <sub>g4</sub>	0.0000	P <sub>g4</sub>	46.0000
P <sub>B4</sub>	0.0000	P <sub>d2</sub>	0.0000	P <sub>d2</sub>	0.0000
P <sub>B5</sub>	-0.0000	P <sub>d3</sub>	100.0000	P <sub>d2</sub>	0.0000
P <sub>1</sub>	1.3000	P <sub>B1</sub>	0.0000	P <sub>B1</sub>	0.0000
P <sub>2</sub>	1.3000	P <sub>B2</sub>	0.0000	P <sub>B2</sub>	0.0000
P <sub>3</sub>	1.3000	P <sub>B3</sub>	0.0000	P <sub>B3</sub>	0.0000
P <sub>4</sub>	1.3000	P <sub>B4</sub>	0.0000	P <sub>B4</sub>	0.0000
		P <sub>B5</sub>	0.0000	P <sub>B5</sub>	0.0000
		$\theta_1$	0.0000	$\theta_1$	0.0000
		$\theta_2$	0.0000	$\theta_2$	0.0000
		$\theta_3$	0.0000	$\theta_3$	0.0000
		$\theta_4$	0.0000	$\theta_4$	0.0000

Lagrange multipliers on the last 4 equality constraints give the improvement in the objective function if we increase the right-hand-side of the corresponding equation by 1 unit. These are the nodal prices, given in \$/per unit-hr. The numbers are all \$1300/per unit-hr or \$13.00/MW-hr. This is also the coefficient of the demand at bus 2, P<sub>d2</sub>.

Now let's consider the Lagrange multipliers (LM):

- Lower bounds: LMs on  $P_{g1}$  and  $P_{d3}$  are non-zero, indicating they are at their lower bounds, as confirmed by decision vector on pg. 20.
- Upper bounds: LMs on  $P_{g2}$  and  $P_{g4}$  are non-zero, indicating they are at their upper bounds, as confirmed by decision vector on pg. 20.
- Not constrained (regulating): Only LMs on  $P_{d2}$  is 0 for both lower and upper bounds, indicating it is not at either bound (this variable is “regulating”).

Observation! There is only ONE unconstrained variable,  $P_{d2}$ , and it is also the variable that is setting the nodal prices (\$13.00/MW-hr) throughout the network! A look at the coefficients will show why:

$$\begin{aligned} s_{g2} &= 1211 \text{ \$/pu-hr} & s_{d2} &= -1300 \text{ \$/pu-hr} \\ s_{g4} &= 1254 \text{ \$/pu-hr} & s_{d3} &= -1200 \text{ \$/pu-hr} \\ s_{g1} &= 1307 \text{ \$/pu-hr} & & \end{aligned}$$

Think of the algorithm like this:

1. *Initialize*: It first sets generation and load at lower limits (there is no choice about this much supply and demand). One variable must come off its lower bound in order to provide power balance. Since sum of load lower bounds is 3, and sum of gen lower bounds is 1.37, one or more of the gens must come off their lower bounds by 1.63 in order to provide a feasible solution. This gen will be the least expensive one(s). In this case, it is G2 and G4 (G2 gets pushed to its limit in this step).

2. *Maximally increase social surplus*: Then it takes a MW of supply and a MW of demand from the gen/load pair that is not at upper bounds and provides the most positive surplus. This will be the gen with the least cost and the load with the greatest utility, as long as surplus is positive. In our example, the first gen/load pair taken, after finding a feasible solution, is G4/D2. It continues to take a MW of supply and a MW of demand from this gen/load pair until one of them hits an upper limit.
3. *Cycle or terminate*: As soon as either the gen or the load of the max-surplus gen/load pair reaches its upper limit, it will cycle back to step 2 and replace the gen or load that just hit its upper limit with the one that yields the next largest surplus. In our case, G4 reaches its upper limit first, and it tries to replace it with G1. But the G1/D2 pair has coefficients that result in a negative surplus! So the maximum surplus is found when G4 reaches its upper limit.

You should be able to see from step 3 that the algorithm will always terminate with just one gen or load regulating (since step 3 cycles or terminates just when one of the gen/load pair hits its limit, leaving the other one still regulating), and that regulating gen or load will set the nodal price throughout the network (for the unconstrained transmission case).

## 5.0 Example: Constrained transmission

We will constrain the transmission on branch 3. Reference to the old solution of Fig. 3, repeated here for convenience, indicates that the flow on branch 3 is 0.1625. So we will constrain that flow to be 0.16.

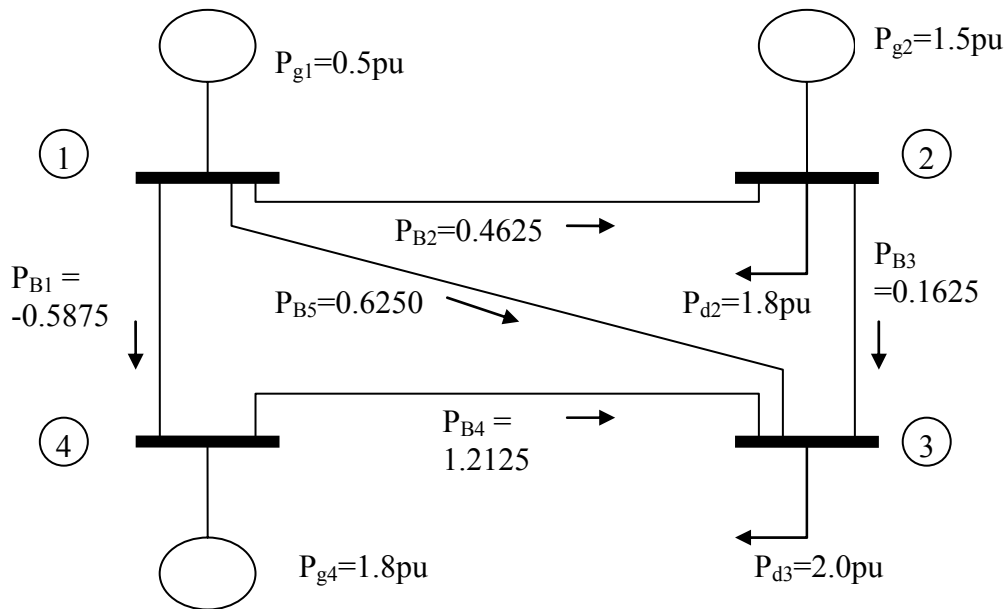


Fig. 3

The Matlab code for this is given below.

```

%Build objective function vector.
c=[1307 1211 1254 -1300 -1200 0 0 0 0 0 0 0 0 0]';

%Build A matrix for inequality constraints Ax<b.
A=[];
%Build b, the right-hand-side of inequality
constraints.
b=[];

%Build Aeq matrix for equality constraints.
Aeq=[0 0 0 0 0 -1 0 0 0 0 10 0 0 -10;
      0 0 0 0 0 0 -1 0 0 0 10 -10 0 0;
      0 0 0 0 0 0 0 -1 0 0 0 10 -10 0;
      0 0 0 0 0 0 0 0 -1 0 0 0 -10 10;
      0 0 0 0 0 0 0 0 0 -1 10 0 -10 0;
      -1 0 0 0 0 0 0 0 0 0 30 -10 -10 -10
      0 -1 0 1 0 0 0 0 0 0 -10 20 -10 0;
      0 0 0 0 1 0 0 0 0 0 -10 -10 30 -10;
      0 0 -1 0 0 0 0 0 0 0 -10 0 -10 20];

%Build right-hand side of equality constraint. It will
be vector of zeros
beq=zeros(9,1);

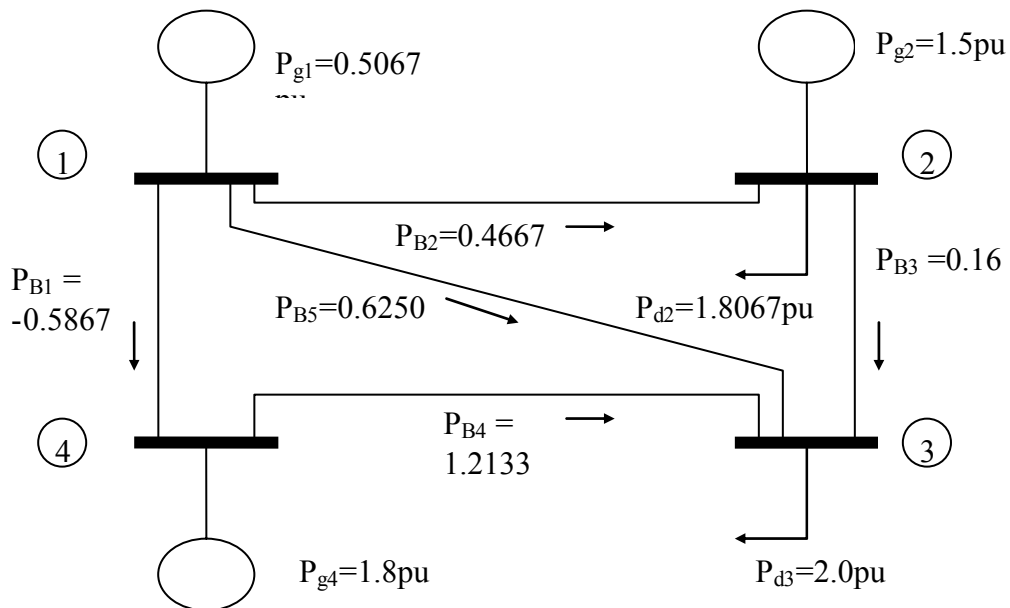
%Build upper and lower bounds on decision variables.
LB=[.50 .375 .45 1 2 -500 -500 -0.16 -500 -500 -
pi -pi -pi -pi]';
UB=[2.00 1.50 1.80 2 3 500 500 0.16 500 500
pi pi pi pi]';
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=linprog(c,A,b,Aeq,beq,L
B,UB);
%'X=', X,FVAL,'equality', LAMBDA.eqlin, 'upper',
LAMBDA.upper, 'lower', LAMBDA.lower
%
% Compute settlements assuming pay-at-bid/offer:
dollars=c.*X;

```

The “new” and the “old” decision vectors are provided below, together with the limits.

New solution		Old solution		Limits		
$P_{g1}$	0.5067	$P_{g1}$	0.5	0.5	$P_{g1}$	2
$P_{g2}$	1.5	$P_{g2}$	1.5	0.375	$P_{g2}$	1.5
$P_{g4}$	1.8	$P_{g4}$	1.8	0.45	$P_{g4}$	1.8
$P_{d2}$	1.8067	$P_{d2}$	1.8	1	$P_{d2}$	2
$P_{d3}$	2.0	$P_{d3}$	2.0	2	$P_{d3}$	3
$P_{B1}$	-0.5867	$P_{B1}$	-0.5875	-500	$P_{B1}$	500
$P_{B2}$	0.4667	$P_{B2}$	0.4625	-500	$P_{B2}$	500
$P_{B3}$	0.1600	$P_{B3}$	0.1625	-0.16	$P_{B3}$	0.16
$P_{B4}$	1.2133	$P_{B4}$	1.2125	-500	$P_{B4}$	500
$P_{B5}$	0.6267	$P_{B5}$	0.6250	-500	$P_{B5}$	500
$\theta_1$	0.0127	$\theta_1$	0.0125	$-\pi$	$\theta_1$	$\pi$
$\theta_2$	-0.0340	$\theta_2$	-0.0338	$-\pi$	$\theta_2$	$\pi$
$\theta_3$	-0.05	$\theta_3$	-0.05	$-\pi$	$\theta_3$	$\pi$
$\theta_4$	0.0713	$\theta_4$	0.0712	$-\pi$	$\theta_4$	$\pi$

The new solution is provided in Fig. 4.



**Objective function value:** The objective function that Matlab provided (FVAL) in the unconstrained case was  $Z=-12.8$  \$/hr (social surplus of \$12.80/hr). Now in the constrained case it is  $Z=-\$12.75$ /hr (social surplus of \$12.75/hr).

The social surplus has decreased, illustrating the principle that adding new constraints can never result in an improvement in the objective function.

**Lagrange multipliers:** The Lagrange multipliers (the same as the dual variables), are given in Table 3.

Table 3: Lagrange multipliers for constrained transm.

Equality constraints		Lower bounds		Upper bounds	
Equation	Value*10 <sup>3</sup>	Variable	value	variable	value
P <sub>B1</sub>	-0.0000	P <sub>g1</sub>	0.0000	P <sub>g1</sub>	0.0000
P <sub>B2</sub>	0.0000	P <sub>g2</sub>	0.0000	P <sub>g2</sub>	89.0000
P <sub>B3</sub>	0.0187	P <sub>g4</sub>	0.0000	P <sub>g4</sub>	55.3333
P <sub>B4</sub>	0.0000	P <sub>d2</sub>	0.0000	P <sub>d2</sub>	0.0000
P <sub>B5</sub>	0.0000	P <sub>d3</sub>	111.6667	P <sub>d3</sub>	0.0000
P <sub>1</sub>	1.3070	P <sub>B1</sub>	0.0000	P <sub>B1</sub>	0.0000
P <sub>2</sub>	1.3000	P <sub>B2</sub>	0.0000	P <sub>B2</sub>	0.0000
P <sub>3</sub>	1.3117	P <sub>B3</sub>	0.0000	P <sub>B3</sub>	18.6667
P <sub>4</sub>	1.3093	P <sub>B4</sub>	0.0000	P <sub>B4</sub>	0.0000
		P <sub>B5</sub>	0.0000	P <sub>B5</sub>	0.0000
		θ <sub>1</sub>	0.0000	θ <sub>1</sub>	0.0000
		θ <sub>2</sub>	0.0000	θ <sub>2</sub>	0.0000
		θ <sub>3</sub>	0.0000	θ <sub>3</sub>	0.0000
		θ <sub>4</sub>	0.0000	θ <sub>4</sub>	0.0000

Some observations:

1. In the unconstrained case, all 4 nodal prices were \$13/MWhr, now, in the constrained case, only bus 2 is \$13/MWhr (set by the  $P_{d2}$  offer), which is a regulating unit. All other nodal prices differ.
2.  $P_{g1}$  is also regulating, and therefore the bus 1 nodal price is set by the  $P_{g1}$  offer which was \$13.07/hr. Notice that here we have two buses regulating. This will not happen in the unconstrained case.
3. Buses 3 and 4 have load or generation at a limit. Bus 3 has  $P_{d3}$  at its lower limit, and bus 4 has  $P_{g4}$  at its upper limit. So neither of these buses are regulating. Notice that the nodal prices at these buses are different from the cost or utility function coefficient at the bus:
  - Bus 3 has utility function coefficient of 1200 whereas its LMP is 1311.70
  - Bus 4 has cost function coefficient of 1254 whereas its LMP is 1309.30.

This shows that buses with regulating units or demands set their own price, whereas non-regulating buses have prices set by other buses.
4. If there were no binding transmission constraints (effectively an infinite transmission capacity situation), then the prices at buses 3 and 4 would be set by one other bus in the network. But with a binding transmission constraint (i.e., presence of congestion), then the prices will be set by the units

needed to supply an additional MW at the bus AND maintain flow within the limit. This will necessarily involve more than one unit.

Comment #4 is worth investigating further.

Let's increase the lower limit of the load at the highest price bus, bus #3, from 2.0 to 2.01 per unit, an increase of 1 MW. The resulting dispatch and flows are shown in Fig. 5. In order to gain intuition into what has happened, we have repeated Fig. 4 just below it so as to provide a convenient comparison.

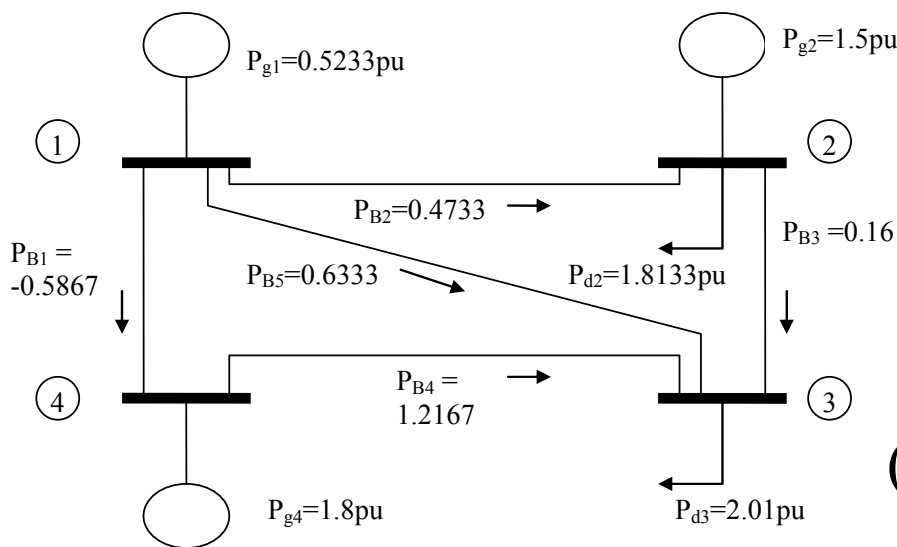


Fig. 5  
 $P_{D3}=2.01$   
 (new dispatch)

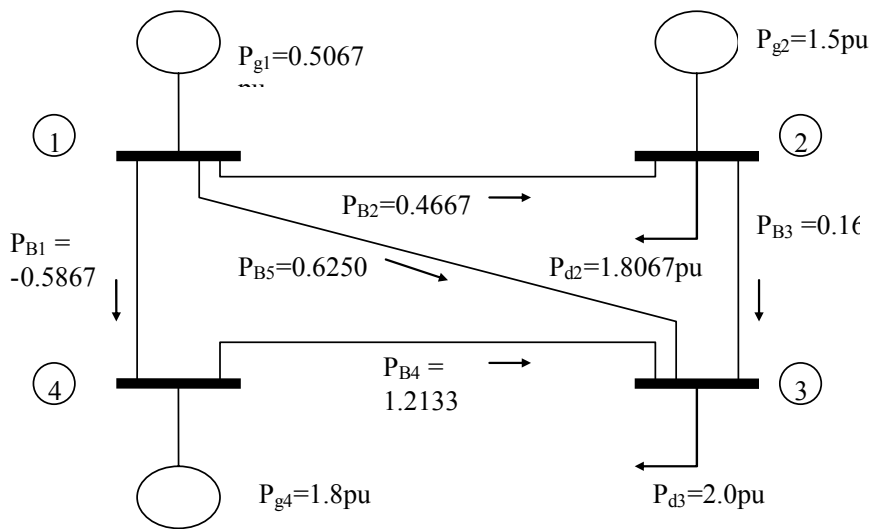


Fig. 4  
 $P_{D3}=2.0$   
 (old dispatch)

The comparison shows that in order to supply an additional MW at bus 3, *the generation levels of 2 different units had to be modified*. Specifically,  $P_{G1}$  was increased from 0.5067 to 0.5233 (+0.0166 pu or 1.66 MW generation), and  $P_{d2}$  was increased from 1.8067 to 1.8133 (+.0066 pu or 0.66 MW load).

So total load increase was the 1 MW from  $P_{D3}$  and the 0.66 MW from  $P_{D2}$  for total of 1.66 MW, which balances with the 1.66 MW generation increase of  $P_{G1}$ .

Question: But why just increase  $P_{G1}$  by 1 MW? Or why not just decrease  $P_{D2}$  by 1 MW?

Answer: Because the resulting flow on branch 3 would exceed its capacity!!!

In fact, it is not possible to supply additional load at bus 3 with only a single unit increase. We will always have to compensate for the load AND redispatch to compensate for the additional flow on the branch 3. As a result, the nodal price at bus 3 is a function of the generation costs at those buses that are used in the particular redispatch that achieves the maximum surplus.

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[1] Midwest ISO Training Materials, "Bids and Offers," 2008, available at [http://www.midwestiso.org/publish/Folder/10b1ff\\_101f945f78e\\_-7b9a0a48324a?rev=1](http://www.midwestiso.org/publish/Folder/10b1ff_101f945f78e_-7b9a0a48324a?rev=1).