Flux-linkage equations for 7-winding representation (similar to eq. 4.11 in text)

$$
\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_D \\ \lambda_Q \\ \lambda_G \end{bmatrix}
=
\begin{bmatrix}
L_{aa} & L_{ab} & L_{ac} & L_{aF} & L_{aD} & L_{aQ} & L_{aG} \\
L_{ba} & L_{bb} & L_{bc} & L_{bF} & L_{bD} & L_{bQ} & L_{bG} \\
L_{ca} & L_{bc} & L_{cc} & L_{cF} & L_{cD} & L_{cQ} & L_{cG} \\
L_{Fa} & L_{Fb} & L_{Fc} & L_{FF} & L_{FD} & L_{FQ} & L_{FG} \\
L_{Da} & L_{Db} & L_{Dc} & L_{DF} & L_{DD} & L_{DQ} & L_{DG} \\
L_{Qa} & L_{Qb} & L_{Qc} & L_{QF} & L_{QD} & L_{QQ} & L_{QG} \\
L_{Ga} & L_{Gb} & L_{Gc} & L_{GF} & L_{GD} & L_{GQ} & L_{GG}
\end{bmatrix}
\begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix}
$$

The above terms are defined as follows:

Stator-stator terms:

$L_{aa}=L_s+L_m\cos 2\theta$
$L_{ab}=-[M_s+L_m\cos 2(\theta+30°)]$
$L_{ac}=-[M_s+L_m\cos 2(\theta+150°)]$

$L_{ba}=-[M_s+L_m\cos 2(\theta+30°)]$
$L_{bb}=L_s+L_m\cos 2(\theta-120°)$
$L_{bc}=-[M_s+L_m\cos 2(\theta-90°)]$

$L_{ca}=-[M_s+L_m\cos 2(\theta+150°)]$
$L_{cb}=-[M_s+L_m\cos 2(\theta-90°)]$
$L_{cc}=L_s+L_m\cos 2(\theta-240°)$

Rotor-rotor terms:

$L_{FF}=L_F$
$L_{FD}=M_R$
$L_{FQ}= L_{FG} =0$

$L_{DF}=M_R$ $L_{QF}=L_{QD}=0$ $L_{GF}=L_{GD}=0$
$L_{DD}=L_D$ $L_{QQ}=L_Q$ $L_{GQ}=M_Y$
$L_{DQ}=L_{DG}=0$ $L_{QG}=M_Y$ $L_{GG}=L_G$

Stator-rotor terms:

$L_{aF}=M_F\cos\theta$
$L_{aD}=M_D\cos\theta$
$L_{aQ}=M_Q\sin\theta$
$L_{aG}=M_G\sin\theta$

$L_{bF}=M_F\cos(\theta-120°)$
$L_{bD}=M_D\cos(\theta-120°)$
$L_{bQ}=M_Q\sin(\theta-120°)$
$L_{bG}=M_G\sin(\theta-120°)$

$L_{cF}=M_F\cos(\theta-240°)$
$L_{cD}=M_D\cos(\theta-240°)$
$L_{cQ}=M_Q\sin(\theta-240°)$
$L_{cG}=M_G\sin(\theta-240°)$

Rotor-Stator terms:

$L_{Fa}=M_F\cos\theta$
$L_{Fb}=M_F\cos(\theta-120°)$
$L_{Fc}=M_F\cos(\theta-240°)$

$L_{Da}=M_D\cos\theta$
$L_{Db}=M_D\cos(\theta-120°)$
$L_{Dc}=M_D\cos(\theta-240°)$

$L_{Qa}=M_Q\sin\theta$
$L_{Qb}=M_Q\sin(\theta-120°)$
$L_{Qc}=M_Q\sin(\theta-240°)$

$L_{Ga}=M_G\sin\theta$
$L_{Gb}=M_G\sin(\theta-120°)$
$L_{Gc}=M_G\sin(\theta-240°)$

So the compact form of the flux linkage equations are

$$\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = [\underline{L}] \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{\underline{L}}_{aa} & \underline{\underline{L}}_{aR} \\ \underline{\underline{L}}_{Ra} & \underline{\underline{L}}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} \qquad \text{(eq. L)}$$

which, when expanded with the expressions for self and mutual inductances, become:

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_D \\ \lambda_Q \\ \lambda_G \end{bmatrix} = \begin{bmatrix} L_S + L_m \cos 2\theta & -[M_S + L_m \cos 2(\theta + 30°)] & -[M_S + L_m \cos 2(\theta + 150°)] & M_F \cos\theta & M_D \cos\theta & M_Q \sin\theta & M_G \sin\theta \\ -[M_S + L_m \cos 2(\theta + 30°)] & L_S + L_m \cos 2(\theta - 120°) & -[M_S + L_m \cos 2(\theta - 90°)] & M_F \cos(\theta - 120°) & M_D \cos(\theta - 120°) & M_Q \sin(\theta - 120°) & M_G \sin(\theta - 120°) \\ -[M_S + L_m \cos 2(\theta + 150°) & -[M_S + L_m \cos 2(\theta - 90°)] & L_S + L_m \cos 2(\theta - 240°) & M_F \cos(\theta - 240°) & M_D \cos(\theta - 240°) & M_Q \sin(\theta - 240°) & M_G \sin(\theta - 240°) \\ M_F \cos\theta & M_F \cos(\theta - 120°) & M_F \cos(\theta - 240°) & L_F & M_R & 0 & 0 \\ M_D \cos\theta & M_D \cos(\theta - 120°) & M_D \cos(\theta - 240°) & M_R & L_D & 0 & 0 \\ M_Q \sin\theta & M_Q \sin(\theta - 120°) & M_Q \sin(\theta - 240°) & 0 & 0 & L_Q & M_Y \\ M_G \sin\theta & M_G \sin(\theta - 120°) & M_G \sin(\theta - 240°) & 0 & 0 & M_Y & L_G \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix}$$

(eq. L-ex)

Voltage equations

Consider the stator circuit appears as in Fig. 1:

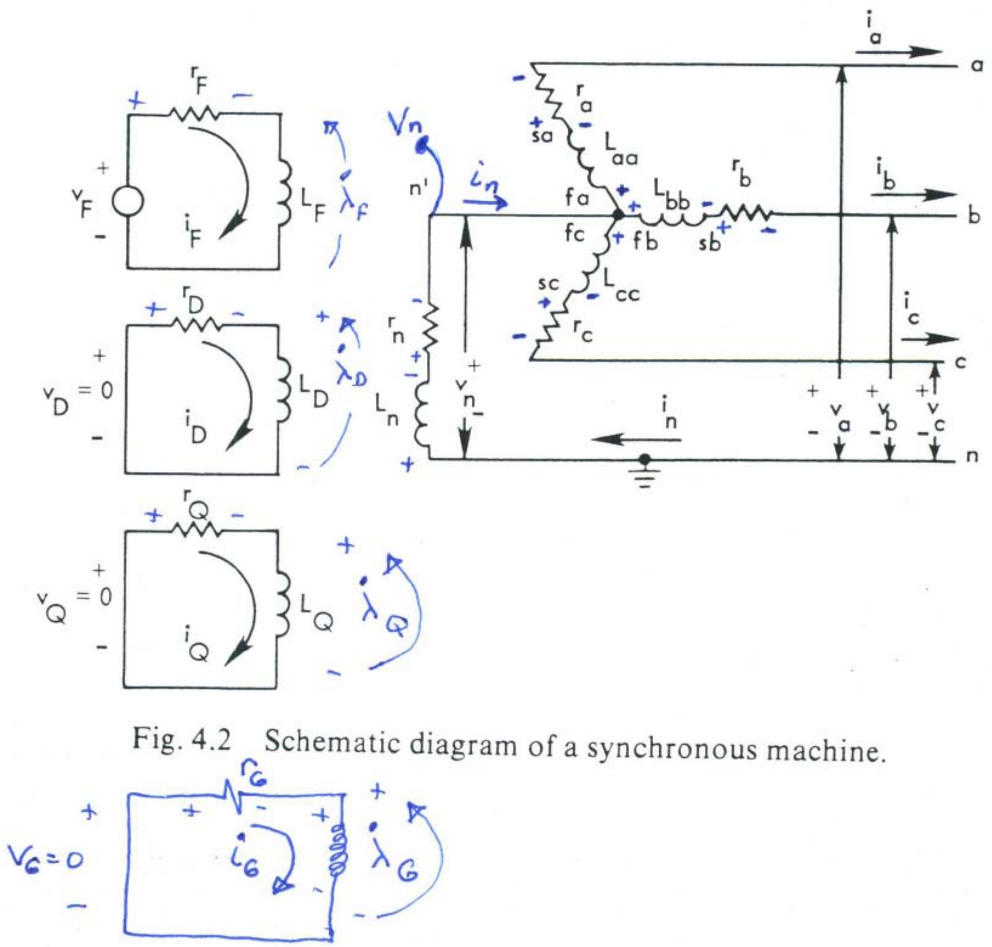


Fig. 4.2 Schematic diagram of a synchronous machine.

Fig. 1

The current direction in the phases, which is out of the terminals for generator operation, produces a flux that is along the negative axis of the respective phase axis.

We assume that the neutral conductor is not coupled with any other circuit.

We can write a voltage equation for each of the phase windings as follows:

$$v_a = -i_a r_a - \dot{\lambda}_a + v_n$$

$$v_b = -i_b r_b - \dot{\lambda}_b + v_n$$

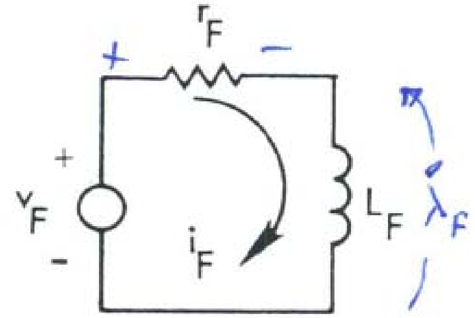
$$v_c = -i_c r_c - \dot{\lambda}_c + v_n$$

We may also write a voltage equation for the neutral circuit as follows:

$$v_n = -i_n r_n - L_n \dot{i}_n = -(i_a + i_b + i_c) r_n - L_n (\dot{i}_a + \dot{i}_b + \dot{i}_c)$$

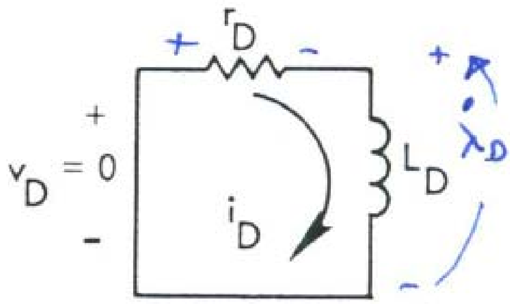
Now let's look at the rotor circuits. There are four of them.

Fig. 2: D-Axis Field



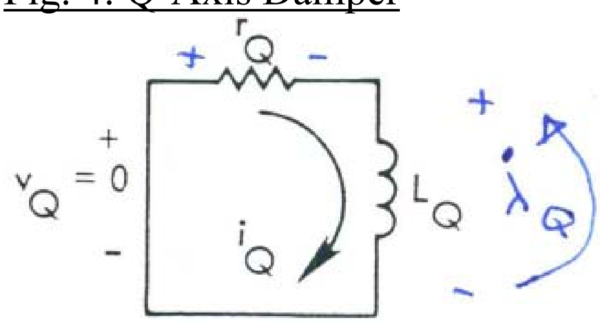
$$v_F = r_F i_F + \dot{\lambda}_F$$
$$\Rightarrow -v_F = -r_F i_F - \dot{\lambda}_F$$

Fig. 3: D-Axis Damper



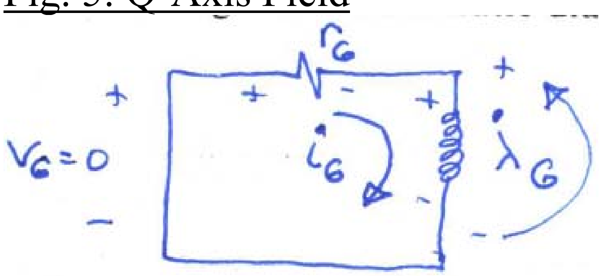
$$0 = -r_D i_D - \dot{\lambda}_D$$

Fig. 4: Q-Axis Damper



$$0 = -r_Q i_Q - \dot{\lambda}_Q$$

Fig. 5: Q-Axis Field



$$0 = -r_G i_G - \dot{\lambda}_G$$

Putting all of these equations together in matrix form, we have that:

$$\begin{bmatrix} v_a \\ v_b \\ v_c \\ -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_G \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix} - \begin{bmatrix} \dot{\lambda}_a \\ \dot{\lambda}_b \\ \dot{\lambda}_c \\ \dot{\lambda}_F \\ \dot{\lambda}_D \\ \dot{\lambda}_Q \\ \dot{\lambda}_G \end{bmatrix} + \begin{bmatrix} v_n \\ v_n \\ v_n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(eq 4.23')}$$

We can write this more compactly, similar to eq. 4.26 in text:

$$\begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FDQG} \end{bmatrix} = -\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} - \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} + \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix} \qquad \text{(eq. 4.26')}$$

Motivation for Park's Transformation

We would like to get the above equation into state-space form (xdot=Ax) so that we can combine it with our inertial equations (and then be able to apply numerical integration and solve them together).

We notice, however, that we have two different types of state variables in the above equations: flux linkages (λ) and currents (i). So we need to eliminate one of them, and this is not hard since we have that flux linkages can be easily expressed as functions of the currents that produce them. For example, for a single conductor, we write that λ=Li (see also first equation in these notes).

But eq. 4.26' has derivatives on λ. Again, no problem, since dλ/dt=d(Li)/dt.

It is here that we run into trouble, since the inductances that we are dealing with are, in general, functions of θ, which is itself a function of time. Therefore the inductances are functions of time, and differentiation of flux linkages results in expressions like:

$$\frac{d\lambda}{dt} = \frac{dL}{dt} i + \frac{di}{dt} L$$

The differentiation with respect to L, dL/dt, will result in a time-varying coefficient on the state variable. When we replace, in eq. 4.26', the derivatives on λ with the derivatives on i, and then solve for the derivatives on i (in order to obtain $\dot{\underline{x}} = \underline{A}\underline{x}$), we will obtain current variables on the right-hand-side that have *time varying coefficients*, i.e., the coefficient matrix A will not be constant. This means that we will have to deal with differential equations with time varying coefficients, which are generally more difficult to solve than differential equations with constant coefficients.

This presents some significant difficulties, in terms of solution, that we would like to avoid. We look for a different approach.

The different approach is based on the observation that our trouble comes from the inductances related to the stator (phase windings):

- Stator self inductances
- Stator-stator mutual inductances
- Stator-rotor mutual inductances

i.e., all of these have time-varying inductances.

In order to alleviate the trouble, we will ***project*** the a-b-c currents onto the D and Q axes.

In making these projections, we want to obtain expressions for the components of the stator currents that are in phase with the D and Q axes.

One can visualize the projection by thinking of the a-b-c currents as having sinusoidal variation IN TIME along their respective axes. The picture below illustrates for the a-phase.

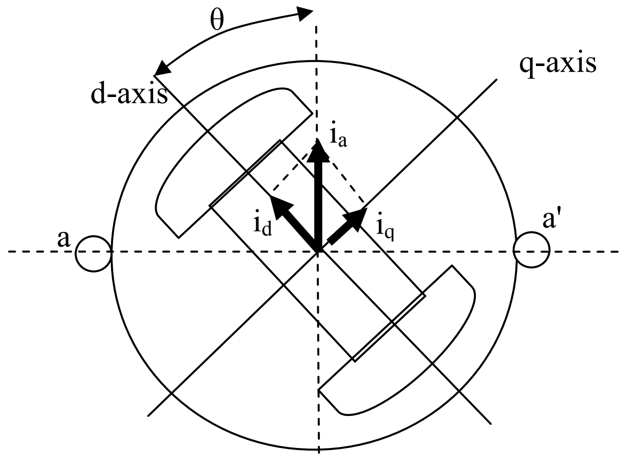


Fig. 6

Decomposing the b-phase currents and the c-phase currents in the same way, and then adding them up, provides us with:

$$i_d = k_d\left(i_a \cos\theta + i_b \cos(\theta - 120°) + i_c \cos(\theta + 120°)\right)$$
$$i_q = k_q\left(i_a \sin\theta + i_b \sin(\theta - 120°) + i_c \sin(\theta + 120°)\right)$$

Here, the constants k_d and k_q are chosen so as to simplify the numerical coefficients in the generalized KVL equations that we will get.

But note: we have transformed 3 variables i_a, i_b, and i_c into two variables i_d and i_q. This yields an undetermined system, meaning

- We can uniquely transform i_a, i_b, and i_c to i_d and i_q
- We cannot uniquely transform i_d and i_q to i_a, i_b, and i_c.

So we need a third current. We take this current proportional to the zero-sequence current:

$$i_0 = k_0\left(i_a + i_b + i_c\right) \qquad \text{(i-zero)}$$

We note that, under balanced conditions, i_0 is zero, and therefore produces no flux at all. In fact, it is possible to show that i_0 produces no flux which links the rotor windings at all (see Concordia's book, pg. 14 and also Kimbark Vol III, pg. 60). The implication is that under all conditions, **i_d and i_q produce the exact same flux as i_a, i_b, and i_c.**

We write our transformation more compactly as:

$$\underbrace{\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}}_{\underline{i}_{odq}} = \underbrace{\begin{bmatrix} k_0 & k_0 & k_0 \\ k_d \cos\theta & k_d \cos(\theta - 120) & k_d \cos(\theta + 120) \\ k_q \sin\theta & k_q \sin(\theta - 120) & k_q \sin(\theta + 120) \end{bmatrix}}_{\underline{P}} \underbrace{\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}}_{\underline{i}_{abc}}$$

$$\underline{i}_{0dq} = \underline{P}\,\underline{i}_{abc} \qquad \text{(eq. 4.3)}$$

We may also operate on the voltages and fluxes in exactly the same way:

$$\underline{v}_{0dq} = \underline{P}\,\underline{v}_{abc}, \qquad \underline{\lambda}_{0dq} = \underline{P}\,\underline{\lambda}_{abc} \qquad \text{(eq. 4.7)}$$

This transformation resulted from the work done by Blondel (1923), Doherty and Nickle (1926), and Park (1929, 1933), and as a result, is usually called "Park's transformation," and the transformation matrix P is usually called "Park's transformation matrix" or just "Park's matrix."

In Park's original paper, he used k_0=1.0 and k_d=1.0, and k_q=-1.0 (he assumed the q-axis as leading the d-axis; if he would have assumed the q-axis as lagging the d-axis, as we have done, then he would have had k_q=1.0). However, there are two main disadvantages with this choice:

1. The transformation is not orthogonal. This means that $\underline{P}^{-1} \neq \underline{P}^T$. If the transformation were orthogonal ($\underline{P}^{-1}=\underline{P}^T$), then the power calculation, which is $p = \underline{v}_{abc}^T \underline{i}_{abc}$, is also given by $p = \underline{v}_{0dq}^T \underline{i}_{0dq}$. This can be proven (see eq. 4.10 in text) since, from eqs. 4.3 and 4.7, $\underline{P}^{-1} \underline{v}_{0dq} = \underline{v}_{abc}$ and $\underline{P}^{-1} \underline{i}_{0dq} = \underline{i}_{abc}$, we may write:
$$p = \underline{v}_{abc}^T \underline{i}_{abc} = \left(\underline{P}^{-1} \underline{v}_{0dq}\right)^T \left(\underline{P}^{-1} \underline{i}_{0dq}\right)$$
Recalling that $(ab)^T=b^Ta^T$, the above is:
$$p = \underline{v}_{abc}^T \underline{i}_{abc} = \underline{v}_{odq}^T \left(\underline{P}^{-1}\right)^T \left(\underline{P}^{-1} \underline{i}_{0dq}\right) = \underline{v}_{odq}^T \left(\underline{P}\right)\left(\underline{P}^{-1} \underline{i}_{0dq}\right)$$
$$= \underline{v}_{odq}^T \underline{i}_{0dq}$$

2. The transformed mutual inductances, when per-unitized, do not provide that $M_{jk}=M_{kj}$, implying that the per-unit inductance matrix is not symmetric. This prevents us from finding a real physical circuit to use in modeling the transformed system.

In order to overcome these problems, we (Anderson and Fouad) make a different choice of constants, according to:
$$k_0 = \frac{1}{\sqrt{3}}, \qquad k_d = k_q = \sqrt{\frac{2}{3}}$$
The choice of k_0, when applied to eq. (i-zero) above, results in:
$$i_0 = \frac{1}{\sqrt{3}}\left(i_a + i_b + i_c\right) = \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}} i_a + \frac{1}{\sqrt{2}} i_b + \frac{1}{\sqrt{2}} i_c\right)$$
So we see that the factor $\sqrt{\frac{2}{3}}$ is the multiplier on all three equations, resulting in a Park's transformation (and the one that we will use) as:

$$\underline{P} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta-120) & \cos(\theta+120) \\ \sin\theta & \sin(\theta-120) & \sin(\theta+120) \end{bmatrix}$$

Park's Transformation Applied toVoltage equations for 7-winding representation

Now perform the Park's transformation on both sides of the voltage equation (eq. 4.23' or 4.26'). Note that we apply $\underline{P}$ to only the a-b-c quantities, i.e., we leave the F-D-Q-G quantities alone (the rotor-rotor quantities are constants and therefore need no transformation) since these quantities are already on the rotor (and the rotor-rotor inductances are already constants). This means we need to multiply eq. (4.23' or 4.26') through by a matrix

$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}$ where $\underline{U}_4$ is a 4x4 identity matrix.

Recall (4.26') is:

$$\begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FDQG} \end{bmatrix} = -\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} - \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} + \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix} \qquad \text{(eq. 4.26')}$$

Multiplying through by our matrix, we obtain:

$$\underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}\begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}\begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}\begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

(eq. tve1)

We need to express eq. (tve1) in terms of 0-d-q quantities. In what follows below, we do this one term at a time. Our general procedure will be to replace the a-b-c quantities with 0-d-q quantities and then simplify.

The easiest one is term 1, so we will begin with it.

Term 1:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_{abc} \\ \underline{v}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{Pv}_{abc} \\ \underline{v}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}$$

Term 2:

This term is:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Note that

$$\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P} & 0 \\ 0 & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}^{-1} & 0 \\ 0 & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Substitution yields:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{P}^{-1} & 0 \\ 0 & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{P}\underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\underline{R}_{abc}\underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Note that the upper left-hand element has a diagonal matrix in the middle of two orthogonal matrices.

Fact: If $\underline{P}$ is orgthogonal, then $\underline{P}\,\underline{R}_{abc}\,\underline{P}^{-1} = \underline{R}_{abc}$ if $\underline{R}_{abc}$ is diagonal having equal elements on the diagonal.

You can test this as follows. Let

$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. It is easy to show this is orthogonal using $\underline{A}\,\underline{A}^T=\underline{U}$.

Then try multiplying $\underline{A}\,\underline{R}\,\underline{A}^T$ where $\underline{R} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

It is easy to prove as follows. If $\underline{R}$ is a diagonal matrix with all of its diagonal elements the same, call them r, then $\underline{R}=r\underline{U}$. Then

$$\underline{ARA}^T=\underline{A}r\underline{UA}^T=r\underline{AUA}^T=r\underline{AA}^T=r\underline{U}=\underline{R}.$$

Here, we will assume $r_a=r_b=r_c$ which is very typical of synchronous machines and simply implies that all phase windings are equal length with the same type of conductor, which is always the case.

Therefore term 2 is just:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{P}\,\underline{R}_{abc}\,\underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Repeating our equation (tve1) here for convenience….

$$\underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_{abc}\\ \underline{v}_{FDQG}\end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FDQG}\end{bmatrix}\begin{bmatrix}\underline{i}_{abc}\\ \underline{i}_{FDQG}\end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\dot{\underline{\lambda}}_{abc}\\ \dot{\underline{\lambda}}_{FDQG}\end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_n\\ \underline{0}\end{bmatrix}}_{\text{term 4}}$$

and recalling what we have done so far:

TERM 1: $$\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_{abc}\\ \underline{v}_{FDQG}\end{bmatrix} = \begin{bmatrix}\underline{Pv}_{abc}\\ \underline{v}_{FDQG}\end{bmatrix} = \begin{bmatrix}\underline{v}_{0dq}\\ \underline{v}_{FDQG}\end{bmatrix}$$

TERM 2:

$$\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FDQG}\end{bmatrix}\begin{bmatrix}\underline{i}_{abc}\\ \underline{i}_{FDQG}\end{bmatrix}$$

$$= \begin{bmatrix}\underline{P}\,\underline{R}_{abc}\underline{P}^{-1} & \underline{0}\\ \underline{0} & \underline{R}_{FDQG}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dq}\\ \underline{i}_{FDQG}\end{bmatrix} = \begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FDQG}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dq}\\ \underline{i}_{FDQG}\end{bmatrix}$$

Substituting, we obtain:

$$\underbrace{\begin{bmatrix}\underline{v}_{0dq}\\ \underline{v}_{FDQG}\end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix}\underline{R}_{abc} & \underline{0}\\ \underline{0} & \underline{R}_{FDQG}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dq}\\ \underline{i}_{FDQG}\end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\dot{\underline{\lambda}}_{abc}\\ \dot{\underline{\lambda}}_{FDQG}\end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix}\underline{P} & \underline{0}\\ \underline{0} & \underline{U}_4\end{bmatrix}\begin{bmatrix}\underline{v}_n\\ \underline{0}\end{bmatrix}}_{\text{term 4}}$$

eq. (tve2)

Now we observe that terms 3 and 4 have variables not in terms of 0-d-q quantities. We work on term 4 next (before term 3) because it is easier.

Term 4:

Observe that $\underline{v}_n=[v_n \quad v_n \quad v_n]^T$. Therefore, when we multiply $\underline{Pv}_n$, we get elements in the second and third rows of $\underline{P}$ being scaled by the same constant (v_n) and then summed. Consider these elements in the second and third rows of $\underline{P}$, below.

$$\underline{P} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta-120) & \cos(\theta+120) \\ \sin\theta & \sin(\theta-120) & \sin(\theta+120) \end{bmatrix}$$

So the product of the second row with v_n, or of the third row and v_n, will include a summation of symmetrical components, which will be zero. So the only non-zero element in $\underline{P}\underline{v}_n$ will be the product of the first row of $\underline{P}$ and $\underline{v}_n$, which is

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} v_n \\ v_n \\ v_n \end{bmatrix} = \frac{3v_n}{\sqrt{3}} \qquad (*)$$

But recall the voltage equation indicates that:

$$v_n = -i_n r_n - L_n \dot{i}_n = -(i_a + i_b + i_c)r_n - L_n(\dot{i}_a + \dot{i}_b + \dot{i}_c) \quad (**)$$

Also, recall that

$$i_0 = \frac{1}{\sqrt{3}}\left(i_a + i_b + i_c\right) \Rightarrow i_a + i_b + i_c = \sqrt{3}i_0 \qquad (***)$$

Substitution of (***) into (**) yields:

$$v_n = -(\sqrt{3}i_0)r_n - L_n(\sqrt{3}\dot{i}_0)$$

and replacing v_n in (*) with this, we have:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{Pv}_n \\ \underline{0} \end{bmatrix} = \begin{bmatrix} 3r_n i_0 - 3L_n \dot{i}_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix} \qquad (*\#)$$

where $\underline{n}_{0dq}$ is the first 3 elements and $\underline{0}$ is the last 4 elements.

Now recall eqt. (tve2), repeated here for convenience:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{v}_n \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

and substitute in eqt. (*#) to obtain

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve3)

And so now the only a-b-c variables remaining are in term 3. So let's work on term 3.

Term 3:
Term 3 is:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} \qquad (4.30')$$

So we need to do two things:

1. Obtain $\underline{P}\dot{\underline{\lambda}}_{abc}$ in terms of the 0-d-q quantities.
2. Express all of term 3 in terms of currents instead of flux linkages.

To begin this task, recall that $\underline{\lambda}_{0dq} = \underline{P}\underline{\lambda}_{abc}$, and take derivatives of both sides. Note in differentiating the right-hand-side, we need to account for the fact that P is time-dependent. Thus:

$$\dot{\underline{\lambda}}_{0dq} = \underline{P}\dot{\underline{\lambda}}_{abc} + \dot{\underline{P}}\underline{\lambda}_{abc}$$

Solving for $\underline{P}\dot{\underline{\lambda}}_{abc}$, we obtain:

$$\underline{P}\dot{\underline{\lambda}}_{abc} = \dot{\underline{\lambda}}_{0dq} - \dot{\underline{P}}\underline{\lambda}_{abc} \qquad (\#)$$

But the right-hand side still has $\underline{\lambda}_{abc}$. We can eliminate this using

$$\underline{\lambda}_{abc} = \underline{P}^{-1}\underline{\lambda}_{odq}$$

Substitution into eq. (#) yields:

$$\underline{P}\dot{\underline{\lambda}}_{abc} = \dot{\underline{\lambda}}_{0dq} - \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{odq} \qquad (4.31)$$

Now we have expressed $\underline{P}\dot{\underline{\lambda}}_{abc}$ in terms of the 0-d-q quantities.
Substitution of eq. (4.31) into eq. (4.30') above yields:

$$\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} - \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}}$$

So we have accomplished our objective 1, which was to obtain $\underline{P}\dot{\underline{\lambda}}_{abc}$ in terms of the 0-d-q quantities. Let's substitute the above equation into eq. (tve3)

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix}\begin{bmatrix} \dot{\underline{\lambda}}_{abc} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve3)

to obtain

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve4)

Now we need to accomplish our objective 2, which is to express all of term 3 in terms of currents instead of flux linkages. To do this, let's investigate terms 3a and 3b one at a time. Let's start with term 3a….

Term 3a:

So term 3a is:

$$\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}$$

Our goal is to see if we can express this in terms of currents, which means we will need to use inductances. Let's start by looking at the same expression but without the derivatives, since we know how to write this using Park's transformation and a-b-c flux linkages. This is:

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P} & 0 \\ 0 & \underline{U}_4 \end{bmatrix}\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FDQG} \end{bmatrix} \qquad \text{(eq. 3a-1)}$$

Now to write eq. (3a-1) in terms of the 0dq/FDQG currents (instead of 0dq/FDQG flux linkages), recall from eq. (L), pg. 2, repeated here for convenience

$$\begin{bmatrix} \underline{\lambda}_{abc} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix}\begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} \qquad \text{(eq. 3a-2)}$$

that the vector of abc/FDQG flux linkages on the right of (eq. 3a-1) is related through the inductance matrix to the abc/FDQG currents.

Now recall that the abc/FDQG currents may be related to the 0dq/FDQG currents using the inverse Park Transformation according to:

$$\begin{bmatrix} \underline{i}_{abc} \\ \underline{i}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix} \qquad \text{(eq. 3a-3)}$$

Substitution of (3a-3) into (3a-2) and then what results into (3a-1), we have

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P} & 0 \\ 0 & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{L}_{aa} & \underline{L}_{aR} \\ \underline{L}_{Ra} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{U}_4 \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Performing the above matrix multiplication, we obtain….

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{P}\,\underline{L}_{aa}\,\underline{P}^{-1} & \underline{P}\,\underline{L}_{aR} \\ \underline{L}_{Ra}\,\underline{P}^{-1} & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Now we need to go through each of these four matrix multiplications. I will here omit the details and just give the results (note also in what follows the definition of additional nomenclature for each of the four submatrices):

Submatrix (1,1):

$$\underline{P}\,\underline{L}_{aa}\,\underline{P}^{-1} = \begin{bmatrix} L_0 & 0 & 0 \\ 0 & L_d & 0 \\ 0 & 0 & L_q \end{bmatrix} \equiv \underline{L}_{0dq}$$

where $L_0=L_S-2M_S$, $L_d=L_S+M_S+(3/2)L_m$, and $L_q=L_S+M_S-(3/2)L_m$.

Submatrix (1,2):

$$\underline{P}\,\underline{L}_{aR} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \end{bmatrix} \equiv \underline{L}_m$$

Submatrix (2,1):

$$\underline{L}_{Ra}\underline{P}^{-1} = \begin{bmatrix} 0 & \sqrt{\frac{3}{2}}M_F & 0 \\ 0 & \sqrt{\frac{3}{2}}M_D & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q \\ 0 & 0 & \sqrt{\frac{3}{2}}M_G \end{bmatrix} \equiv \underline{L}_m^T$$

Submatrix (2,2) (note that this submatrix is unchanged from the original inductance matrix):

$$\underline{L}_{RR} = \begin{bmatrix} L_F & M_R & 0 & 0 \\ M_R & L_D & 0 & 0 \\ 0 & 0 & L_Q & M_Y \\ 0 & 0 & M_Y & L_G \end{bmatrix} \equiv \underline{L}_{RR}$$

Using the defined nomenclature above for the 4 elements, we finally have:

$$\begin{bmatrix} \underline{\lambda}_{0dq} \\ \underline{\lambda}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}$$

Expanding…

$$
\begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \lambda_F \\ \lambda_D \\ \lambda_Q \\ \lambda_G \end{bmatrix} =
\begin{bmatrix}
L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_d & 0 & \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 \\
0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \\
0 & \sqrt{\frac{3}{2}}M_F & 0 & L_F & M_R & 0 & 0 \\
0 & \sqrt{\frac{3}{2}}M_D & 0 & M_R & L_D & 0 & 0 \\
0 & 0 & \sqrt{\frac{3}{2}}M_Q & 0 & 0 & L_Q & M_Y \\
0 & 0 & \sqrt{\frac{3}{2}}M_G & 0 & 0 & M_Y & L_G
\end{bmatrix}
\begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix}
\tag{4.20'}
$$

Compare this to eq. (L-ex) on page 2 to see big improvement in simplicity.

<u>Aside</u>: It is convenient here to note from the above matrix relation that λ_d and λ_q are given by:

$$
\underline{\lambda}_{0dq} = \underline{L}_{0dq}\underline{i}_{0dq} + \underline{L}_m \underline{i}_{FDQG} \Rightarrow \lambda_d = L_d i_d + \sqrt{\frac{3}{2}}M_F i_F + \sqrt{\frac{3}{2}}M_D i_D
$$

$$
\Rightarrow \lambda_q = L_q i_q + \sqrt{\frac{3}{2}}M_Q i_Q + \sqrt{\frac{3}{2}}M_G i_G
$$

We will use this in developing term 3b below.

One nice surprise from the above is that THE MATRIX IS CONSTANT!!!

As a result of this "nice surprise," we may differentiate both sides to get:

$$\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix} = \begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FDQG} \end{bmatrix} \qquad (\$)$$

or, again, when expanded, is:

$$\begin{bmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_d \\ \dot{\lambda}_q \\ \dot{\lambda}_F \\ \dot{\lambda}_D \\ \dot{\lambda}_Q \\ \dot{\lambda}_G \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 \\ 0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \\ 0 & \sqrt{\frac{3}{2}}M_F & 0 & L_F & M_R & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}M_D & 0 & M_R & L_D & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q & 0 & 0 & L_Q & M_Y \\ 0 & 0 & \sqrt{\frac{3}{2}}M_G & 0 & 0 & M_Y & L_G \end{bmatrix} \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_Q \\ \dot{i}_G \end{bmatrix}$$

Substitution of ($) for term 3a into eq. (tve4), repeated here for convenience,

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \dot{\underline{\lambda}}_{0dq} \\ \dot{\underline{\lambda}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve4)

results in

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve5)

We are almost done! The only remaining term which contains flux linkages is term 3b.

Term 3b:

Recalling term 3b is: $\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}$

we see that we need to expand the product $\dot{\underline{P}}\underline{P}^{-1}$. First, recall that:

$$\underline{P} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \\ \sin\theta & \sin(\theta - 120) & \sin(\theta + 120) \end{bmatrix}$$

Also, recall that

$$\theta = \omega_{\text{Re}} t + \delta(t) + \pi/2 \;\; ➔ \;\; \dot{\theta} = \omega_{\text{Re}} + \dot{\delta}(t)$$

And note carefully that P is a function of time because the angle θ is a function of t. Therefore we need to differentiate P. This is not hard and results in:

$$\dot{\underline{P}} = \frac{d\underline{P}}{dt} = \sqrt{\frac{2}{3}}\omega \begin{bmatrix} 0 & 0 & 0 \\ -\sin\theta & -\sin(\theta - 120) & -\sin(\theta + 120) \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \end{bmatrix}$$

Now taking the product $\dot{\underline{P}}\underline{P}^{-1}$, we obtain:

$$\dot{\underline{P}}\underline{P}^{-1} = \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}\omega \begin{bmatrix} 0 & 0 & 0 \\ -\sin\theta & -\sin(\theta - 120) & -\sin(\theta + 120) \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta \\ \frac{1}{\sqrt{2}} & \cos(\theta - 120) & \sin(\theta - 120) \\ \frac{1}{\sqrt{2}} & \cos(\theta + 120) & \sin(\theta + 120) \end{bmatrix}$$

$$= \frac{2}{3}\omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3/2 \\ 0 & 3/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

Note in the above that row 1 is all zeros because row 1 in $\dot{P}$ is all zeros. On the other hand, column 1 is all zeros because the multiplication of rows 2 and 3 in $\dot{P}$ by column 1 of $\underline{P}^{-1}$ yield a sum of symmetrical terms.

This provides that:

$$\underline{\dot{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega\lambda_q \\ \omega\lambda_d \end{bmatrix}$$

Some comments on speed voltages $-\omega\lambda_q$ and $\omega\lambda_d$.

- These *speed voltages* together account for the voltages induced in the (fixed) phase windings as a result of the spatially-moving magnetic field from the rotor.
- They represent the fact that a flux wave rotating in synchronism with the rotor will create voltages in the stationary armature coils.
- Speed voltages are so named to contrast them from what may be called *transformer voltages,* which are induced as a result of a time varying magnetic field.
- You may have run across the concept of "speed voltages" in Physics, where you computed a voltage induced in a coil of wire as it moved through a static magnetic field, in which case, you may have used the equation Blv where B is flux density, l is conductor length, and v is the component of the velocity of the moving conductor (or moving field) that is normal with respect to the field flux direction (or conductor).

- The first speed voltage term, -$\omega\lambda_q$, appears in the v_d equation. The second speed voltage term, $\omega\lambda_d$, appears in the v_q equation. Thus, we see that the q-axis flux causes a speed voltage in the d-axis winding, and the d-axis flux causes a speed voltage in the q-axis winding.

Now we are in a position to obtain term 3b. Using the expressions for λ_d and λ_q obtained in the "Aside" of page 20 above, we get:

$$\begin{bmatrix} \underline{\dot{P}P^{-1}\lambda_{0dq}} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega\lambda_q \\ \omega\lambda_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega L_q i_q - \omega\sqrt{\frac{3}{2}}M_Q i_Q - \omega\sqrt{\frac{3}{2}}M_G i_G \\ \omega L_d i_d + \omega\sqrt{\frac{3}{2}}M_F i_F + \omega\sqrt{\frac{3}{2}}M_D i_D \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{speed} \\ \underline{0} \end{bmatrix} \quad (\&)$$

where

$$\underline{speed} = \begin{bmatrix} 0 \\ -\omega L_q i_q - \omega\sqrt{\frac{3}{2}}M_Q i_Q - \omega\sqrt{\frac{3}{2}}M_G i_G \\ \omega L_d i_d + \omega\sqrt{\frac{3}{2}}M_F i_F + \omega\sqrt{\frac{3}{2}}M_D i_D \end{bmatrix}; \quad \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now recalling eq. (tve5),

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = \underbrace{-\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} \underbrace{-\begin{bmatrix} \underline{L}_{0dq} & \underline{L}_m \\ \underline{L}^T{}_m & \underline{L}_{RR} \end{bmatrix}\begin{bmatrix} \underline{\dot{i}}_{0dq} \\ \underline{\dot{i}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \underline{\dot{P}}\underline{P}^{-1}\underline{\lambda}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve5)

we substitute (&) to obtain:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dq} \\ \underline{v}_{FDQG} \end{bmatrix}}_{\text{term 1}} = -\underbrace{\begin{bmatrix} \underline{R}_{abc} & \underline{0} \\ \underline{0} & \underline{R}_{FDQG} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dq} \\ \underline{i}_{FDQG} \end{bmatrix}}_{\text{term 2}} - \underbrace{\begin{bmatrix} \underline{L}_{0dq} & \underline{L}_{m} \\ \underline{L}^{T}{}_{m} & \underline{L}_{RR} \end{bmatrix}\begin{bmatrix} \dot{\underline{i}}_{0dq} \\ \dot{\underline{i}}_{FDQG} \end{bmatrix}}_{\text{term 3a}} + \underbrace{\begin{bmatrix} \underline{speed} \\ \underline{0} \end{bmatrix}}_{\text{term 3b}} + \underbrace{\begin{bmatrix} \underline{n}_{0dq} \\ \underline{0} \end{bmatrix}}_{\text{term 4}}$$

eq. (tve6)

Putting it all together:

Let's re-write the voltage equation eq. (tve6) by substituting in complete expressions for all vectors and submatrices in terms 1, 2, 3a, 3b, and 4, as obtained above:

Term 1 Term 2 Term 3a

$$\begin{bmatrix} v_0 \\ v_d \\ v_q \\ -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r_a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_G \end{bmatrix}\begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix} - \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 \\ 0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \\ 0 & \sqrt{\frac{3}{2}}M_F & 0 & L_F & M_R & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}M_D & 0 & M_R & L_D & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q & 0 & 0 & L_Q & M_Y \\ 0 & 0 & \sqrt{\frac{3}{2}}M_G & 0 & 0 & M_Y & L_G \end{bmatrix}\begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_Q \\ \dot{i}_G \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ -\omega L_q i_q - \omega\sqrt{\frac{3}{2}}M_Q i_Q - \omega\sqrt{\frac{3}{2}}M_G i_G \\ \omega L_d i_d + \omega\sqrt{\frac{3}{2}}M_F i_F + \omega\sqrt{\frac{3}{2}}M_D i_D \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3r_n i_0 - 3L_n \dot{i}_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Term 3b Term 4

Now, observe that each of the non-zero elements of term 3b and term 4 is multiplied by a current or current derivative, and that terms 2 and 3a both get multiplied by vectors of currents or current derivatives, respectively. Therefore, we may "fold-in" Term 3b and Term 4 into the Terms 2 and 3a by combining parts of the non-zero term 3b and 4 elements with the appropriate matrix element in terms 2 and 3a.

For example, we may fold in the $-\omega L_q i_q$ term in row 2 of term 3b by including ωL_q in row 2 (since we are dealing with the second equation), column 3 (since we need the term that multiplies i_q) of term 2. Note that since term 2 has a "minus" sign out front, we do not include the "minus" sign of $-\omega L_q i_q$ when we fold it in. The circle and arrow above illustrate this folding-in operation.

The complete results of all fold-in operations are provided in what follows:

$$\begin{bmatrix} v_0 \\ v_d \\ v_q \\ -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r_a + 3r_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_b & \omega L_q & 0 & 0 & \omega\sqrt{\frac{3}{2}}M_Q & \omega\sqrt{\frac{3}{2}}M_G \\ 0 & -\omega L_D & r_c & -\omega\sqrt{\frac{3}{2}}M_F & -\omega\sqrt{\frac{3}{2}}M_D & 0 & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_G \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix}$$

$$-\begin{bmatrix} L_0 + 3L_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 \\ 0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \\ 0 & \sqrt{\frac{3}{2}}M_F & 0 & L_F & M_R & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}M_D & 0 & M_R & L_D & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}M_Q & 0 & 0 & L_Q & M_Y \\ 0 & 0 & \sqrt{\frac{3}{2}}M_G & 0 & 0 & M_Y & L_G \end{bmatrix} \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_Q \\ \dot{i}_G \end{bmatrix}$$

It is of interest to rearrange the ordering of the variables so that the voltage equations for all d-axis windings are together and the voltage equations for all q-axis windings are together because this will emphasize the presence or absence of the various couplings that we have. The result of this re-ordering of the variables is as follows:

$$
\begin{bmatrix} v_0 \\ v_d \\ -v_F \\ v_D = 0 \\ v_q \\ v_Q = 0 \\ v_G = 0 \end{bmatrix} = -\left[\begin{array}{c|ccc|ccc} r+3r_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & r & 0 & 0 & \omega L_q & \omega\sqrt{\frac{3}{2}}M_Q & \omega\sqrt{\frac{3}{2}}M_G \\ 0 & 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_D & 0 & 0 & 0 \\ \hline 0 & -\omega L_D & -\omega\sqrt{\frac{3}{2}}M_F & -\omega\sqrt{\frac{3}{2}}M_D & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_G \end{array}\right] \begin{bmatrix} i_0 \\ i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix}
$$

$$
-\left[\begin{array}{c|ccc|ccc} L_0+3L_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & L_d & \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}M_F & L_F & M_R & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}M_D & M_R & L_D & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & L_Q & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}M_Q & L_Q & M_Y \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}M_G & M_Y & L_G \end{array}\right] \begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \dot{i}_G \end{bmatrix}
$$

(eq. 4.39')

Some observations about the transformed voltage equations:

1. The first matrix gives
   a. Resistive voltage drops
   b. Speed voltage drops, svd (terms with ω). These svd's
      - Occur in the d- and q- circuits, to represent the fact that a flux wave rotating in synchronism with the rotor will create voltages in the stationary armature coils
      - Do not occur in circuits physically located on the rotor, since there is no motion between the rotating flux wave and the rotor windings.
      - Are caused by currents in the field windings of the "other" axis:
        → the d-circuit svd is caused by i_q, i_Q, and i_G
        → the q-circuit svd is caused by i_d, i_D, and i_F
2. The matrices are *almost* constant, except for the svd terms in the first matrix, but even these terms are practically constant since we only see small changes in ω. The constancy of the matrices is the main motivation behind the Park's transformation.
3. The variables have been reorganized so that all d-axis circuits are together and all q-axis circuits are together. This makes it easy to observe any coupling/decoupling between different sets of circuits.
4. The second matrix gives voltage induced by current (or flux) variation. Note that there is no coupling between the d-axis circuits (d, F, D) and the q-axis circuits (q, Q, G). This is because these two sets of circuits are orthogonal.

Finally, some comments about the Park's transformation:

1. i_d and i_q are currents in a fictitious pair of windings *fixed on the rotor*.
2. These currents produce the same flux as do the a,b,c currents.
3. For balanced steady-state operating conditions, we can use $\underline{i}_{0dq} = \underline{Pi}_{abc}$ to show that the currents in the d and q windings are dc! The implication of this is that:

- The a,b,c currents fixed in space, varying in time **produce the same synchronously rotating magnetic field as**
- The d,q currents, varying in space, fixed in time!

From Kimbark, Vol. III:

Physical interpretation of Park's variables. A physical interpreta tion of the new variables is now in order. The m.m.f. of each arma ture phase, being sinusoidally distributed in space, may be representec by a vector the direction of which is that of the phase axis and th magnitude of which is proportional to the instantaneous phase cur rent. The combined m.m.f. of the three phases may likewise be rep resented by a vector which is the vector sum of the phase-m.m.f vectors. The projections of the combined-m.m.f. vector on the direc and quadrature axes of the field are equal to the sums of the projec tions of the phase-m.m.f. vectors on the respective axes as given b the expressions for i_d and i_q, eqs. 106. The constant $\frac{2}{3}$ is arbitrary Thus i_d may be interpreted as the instantaneous current in a fictitiou armature winding which rotates at the same speed as the field winding and remains in such position that its axis always coincides with the direct axis of the field, the value of the current in this winding being such that it gives the same m.m.f. on this axis as do the actual three instantaneous armature phase currents flowing in the actual armature windings. The interpretation of i_q is similar to that of i_d except tha it acts in the quadrature axis instead of in the direct axis. The i_0 o the new variables is identical with the usual zero-sequence current except that it is an instantaneous value and is defined in terms of the instantaneous phase currents. This current gives no space-fundamental air-gap flux.

The flux linkages of the fictitious armature windings in which i_d and i_q flow are ψ_d and ψ_q, respectively.

In view of the foregoing interpretation of i_d and i_q, it is apparent that their m.m.f.'s are stationary with respect to the rotor and therefore act on paths of constant permeance. Hence the corresponding inductances L_d and L_q are independent of rotor position.

The fictitious direct-axis stator winding and the field winding are inductively coupled. Each has a self-inductance (L_d and L_{ff}), and there is a mutual inductance between them. It should be noted that the mutual inductance has different values in eqs. 116*a* and *d*, being M_f in one and $\frac{3}{2}M_f$ in the other. The difference could have been avoided by a different choice of the constant coefficients in eqs. 105 and 106; however, we will retain the form of the variables given by Park.

Another interesting paragraph from Kimbark Vol. III

In the interpretation of eqs. 116, it was suggested that i_d and i_q were the currents in fictitious *rotating* stator windings — if that paradoxical expression may be used — which gave the same m.m.f.'s as did the actual armature currents in the actual armature windings. But it is not necessary to have the fictitious windings rotate. The same effect can be achieved by conceiving the armature winding to be stationary (as it actually is) and to be a closed-circuit winding with a commutator on which rest brushes which rotate with the field. The magnetic axis of the stator will always coincide with the brush axis. Thus i_d may be interpreted as the current entering and leaving the armature through a pair of brushes which are aligned with the direct axis of the field. Similarly, i_q may be regarded as the current entering and leaving the armature through a second pair of brushes, aligned with the quadrature axis of the field. In other words, the armature may be thought of as like that of a synchronous converter, having both commutator and slip rings, but having brushes in both axes instead of in the quadrature axis only. The actual phase currents, entering the slip rings, give the same m.m.f.'s as do the substitute currents i_d and i_q entering the commutator brushes.

This physical picture is also correct with respect to the voltages. The terms $-\omega\psi_q$ and $\omega\psi_d$ occurring in eqs. 121 and 122 may be regarded as components of applied voltage required to balance the *speed voltages*. The speed voltage across each pair of brushes is proportional to the flux on the axis 90° ahead of the brush axis.