Linearization of Generator Current-State Space Model

We developed a state-space current model for the synchronous machine with the G-circuit represented (see notes on per-unitization), and it was found to be:

$$
\begin{bmatrix} v_d \\ -v_F \\ 0 \\ v_q \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r & 0 & 0 & \omega L_q & \omega k M_Q & \omega k M_G \\ 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 \\ -\omega L_d & -\omega k M_F & -\omega k M_D & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & r_G \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix}
- \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_Q & kM_G \\ 0 & 0 & 0 & kM_Q & L_Q & M_Y \\ 0 & 0 & 0 & kM_G & M_Y & L_G \end{bmatrix} \begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \dot{i}_G \end{bmatrix}
\tag{L-1}
$$

We called this equation 4.74'.

Note the presence of the derivatives on the right-hand-side.

The torque equation was also given as (see notes called "TorqueEquation"):

$$\dot{\omega} = \frac{T_m}{\tau_j} + \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} & \frac{-D}{\tau_j} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \\ \omega \end{bmatrix}$$

(L-2)

and finally, we had

$$\dot{\delta} = \omega - 1 \qquad \text{(L-3)}$$

Note that the state variables are i_d, i_F, i_D, i_q, i_Q, i_G ω, and δ, i.e.,

$$\underline{x} = \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \\ \omega \\ \delta \end{bmatrix} \qquad \text{(L-4)}$$

So we have 8 state variables and 8 equations.

We want to develop a model that will allow us to investigate the stability properties of the system represented by these equations using linear system theory, because the theory of linear systems is so powerfully well-developed, as we have seen in our previous work on eigenvalues, eigenvectors, mode shape, and participation factors.

However, in order to employ linear system theory, the state variables must only appear in our equations in the form of linear terms; that is, the state variables may only appear as a term like $a_i x_i$ where a_i is a constant.

Our equations have some state variables appearing as product terms of the form $x_i x_j$. For example, note in eq. 4.74' (L-1) the appearance of ω in the first matrix, which means that it will get multiplied by a current state variable to result in a term like $\omega L_q i_q$. Also, note that almost every term in the torque equation is the product of two current state variables.

So our current-state-space model is highly non-linear. The process of *linearization* is where we convert the equations so that they contain only linear terms. To do this, we will focus on each equation and convert each nonlinear term to a linear term.

Nothing is free. In order to linearize, we will need to select a "point" in the state-space. Then the price we will pay for the simplification is that the resulting model will only be valid for a region "close to" that selected point.

Our general strategy is based on the recognition that each of our equations in 4.74' look like

$$v = f(\underline{x}, \underline{\dot{x}}) \qquad \text{(L-5)}$$

We will select our point of linearization as $\underline{x}_0$ and then we will develop a model based on the idea that

- Only changes in the state variables may occur (e.g., no changes in parameters)
- The changes in the state variables must be small.

We will denote the changes as:

$$\underline{x} = \underline{x}_0 + \underline{\Delta x} \qquad \text{(L-6)}$$

Differentiating, we also have that:

$$\underline{\dot{x}} = \underline{\dot{x}}_0 + \underline{\Delta \dot{x}} \qquad \text{(L-7a)}$$

The term $\dot{\underline{x}}_0$ in eq. (L-7a) should not be understood as an attempt to take a set of derivatives on a set of constants (which would be zero), but rather a set of derivatives on $\underline{x}$ that are subsequently evaluated at $\underline{x}_0$. That is,

$$\dot{\underline{x}} = \dot{\underline{x}}\Big|_{\underline{x}=\underline{x}_0} + \underline{\Delta\dot{x}} \qquad \text{(L-7b)}$$

The effect of changes in the state variables given by eqs. (L-6) and (L-7a) on eq. (L-5) can be expressed by substituting eqs. (L-6) and (L-7a) into eq. (L-5) and adjusting the right-hand-side of eq. (L-5) accordingly. That is:

$$v_0 + \Delta v = f(\underline{x}_0 + \Delta\underline{x}, \dot{\underline{x}}_0 + \Delta\dot{x}) \qquad \text{(L-8)}$$

Therefore, every time we see a state variable, we will replace it with $x_0 + \Delta x$, and every time we see a state-variable derivative, we will replace it with $\dot{x}_0 + \Delta\dot{x}$.

Once this is done, however, the terms corresponding to products of state variables, i.e., x_1x_2, will appear like

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) \qquad \text{(L-9)}$$

(Fortunately, we have no products of terms containing derivatives of state variables).

Eq. (L-9) is evaluated in the text (pg 208) as:

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) = x_{10}x_{20} + x_{10}\Delta x_2 + x_{20}\Delta x_2 + \Delta x_1\Delta x_2 \text{(L-10)}$$

where it is argued that $\Delta x_1 \Delta x_2$ is negligible and therefore

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) = x_{10}x_{20} + x_{10}\Delta x_2 + x_{20}\Delta x_2 \qquad \text{(L-11)}$$

Noting that

- the left-hand-side of eq. (L-11) gives the product term under the changed condition and
- the first term on the right-hand side gives the product term for the unchanged condition,

we see that the change in the product term is given by:

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) - x_{10}x_{20} = x_{10}\Delta x_2 + x_{20}\Delta x_1 \qquad \text{(L-12)}$$

We will repeatedly employ this strategy of substituting and then recognizing on the right-hand-side the "unchanged" expression. Before we go through this procedure, it is worthwhile to note that the process of linearization comes about more formally through the Taylor series expansion (TSE) about a selected point.

Performing a TSE about the point $\underline{x}_0$, we have:

$$f(\underline{x}_0 + \Delta\underline{x}) = f(\underline{x}_0) + f'(\underline{x}_0)\Delta\underline{x} + \frac{1}{2!} f''(\underline{x}_0)\Delta\underline{x}^2 + \ldots \quad \text{(L-13)}$$

Neglecting the higher-order terms with more than one derivative (justified by the idea that the higher order powers of $\Delta\underline{x}$ are very small), we have that

$$f(\underline{x}_0 + \Delta\underline{x}) \approx f(\underline{x}_0) + f'(\underline{x}_0)\Delta\underline{x}. \quad \text{(L-14)}$$

<u>For example</u>, consider $f(x_1,x_2)=x_1x_2$, which is just a function with a single product term (which was what (L-12) was derived to address).

The TSE of $f(x_1,x_2)=x_1x_2$ about the point $\underline{x}_0=[x_{10}, x_{20}]^T$ is"

$$f(\underline{x}_0 + \Delta\underline{x}) \approx x_{10}x_{20} + \begin{bmatrix} x_{20} & x_{10} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = x_{10}x_{20} + x_{20}\Delta x_1 + x_{10}\Delta x_2$$

where we see that

$$\begin{aligned} & f(\underline{x}_0 + \Delta\underline{x}) - f(\underline{x}_0) \\ &= (x_{10}x_{20} + x_{20}\Delta x_1 + x_{10}\Delta x_2) - (x_{10}x_{20}) \\ &= x_{20}\Delta x_1 + x_{10}\Delta x_2 \end{aligned}$$

which is in agreement with (L-12).

The eight state equations must be linearized one at a time. We will linearize just one of them, for purposes of illustration. We will take the first voltage equation, the one for v_d. It has been lifted out of the equation 4.74' (or L-1 above), and is repeated below for convenience.

$$v_d = -ri_d - \omega L_q i_q - \omega kM_Q i_Q - \omega kM_G i_G - L_d \dot{i}_d - kM_F \dot{i}_F - kM_D \dot{i}_D \tag{L-15}$$

Substituting

$v_d \leftarrow v_{d0} + \Delta v_d$

$i_d \leftarrow i_{d0} + \Delta i_d$, $\dot{i}_d \leftarrow \dot{i}_{d0} + \Delta \dot{i}_d$

$i_q \leftarrow i_{q0} + \Delta i_q$

$i_Q \leftarrow i_{Q0} + \Delta i_Q$

$i_G \leftarrow i_{G0} + \Delta i_G$

$\dot{i}_F \leftarrow \dot{i}_{F0} + \Delta \dot{i}_F$

$\dot{i}_D \leftarrow \dot{i}_{D0} + \Delta \dot{i}_D$

$\omega \leftarrow \omega_0 + \Delta\omega$

we have:

$$\begin{aligned} & v_{d0} + \Delta v_d \\ & = -r(i_{d0} + \Delta i_d) - (\omega_0 + \Delta\omega) L_q (i_{q0} + \Delta i_q) \\ & - (\omega_0 + \Delta\omega) kM_Q (i_{Q0} + \Delta i_Q) \\ & - (\omega_0 + \Delta\omega) kM_G (i_{G0} + \Delta i_G) \\ & - L_d (\dot{i}_{d0} + \Delta \dot{i}_d) - kM_F (\dot{i}_{F0} + \Delta \dot{i}_F) - kM_D (\dot{i}_{D0} + \Delta \dot{i}_D) \end{aligned} \tag{L-16}$$

Recall eq. (L-11) which is the linearized expression for products, repeated below for convenience:

$$(x_{10} + \Delta x_1)(x_{20} + \Delta x_2) = x_{10}x_{20} + x_{10}\Delta x_2 + x_{20}\Delta x_2$$

We apply this equation to each of the product terms in eq. (L-16):

$$\begin{aligned}
&v_{d0} + \Delta v_d \\
&= -r\left(i_{d0} + \Delta i_d\right) - L_q\left(\omega_0 i_{q0} + \omega_0 \Delta i_q + \Delta\omega\, i_{q0}\right) \\
&- kM_Q\left(\omega_0 i_{Q0} + \omega_0 \Delta i_Q + i_{Q0}\Delta\omega\right) \\
&- kM_G\left(\omega_0 i_{G0} + \omega_0 \Delta i_G + i_{G0}\Delta\omega\right) \\
&- L_d\left(\dot{i}_{d0} + \Delta\dot{i}_d\right) - kM_F\left(\dot{i}_{F0} + \Delta\dot{i}_F\right) - kM_D\left(\dot{i}_{D0} + \Delta\dot{i}_D\right)
\end{aligned} \tag{L-17}$$

Separating on the right-hand-side initial condition terms (with zero subscript) from "deviation" terms (with Δ in front), we have:

$$\begin{aligned}
&v_{d0} + \Delta v_d \\
&= \left[-r i_{d0} - L_q\omega_0 i_{q0} - kM_Q\omega_0 i_{Q0} - kM_G\omega_0 i_{G0} - L_d\dot{i}_{d0} - kM_F\dot{i}_{F0} - kM_D\dot{i}_{D0}\right] \\
&+ \Big\{-r\Delta i_d - L_q\left(\omega_0\Delta i_q + \Delta\omega i_{q0}\right) - kM_Q\left(\omega_0\Delta i_Q + i_{Q0}\Delta\omega\right) \\
&\quad - kM_G\left(\omega_0\Delta i_G + i_{G0}\Delta\omega\right) - L_d\Delta\dot{i}_d - kM_F\Delta\dot{i}_F - kM_D\Delta\dot{i}_D\Big\}
\end{aligned} \tag{L-18}$$

Recognizing that the term in square brackets is just v_{d0}, we subtract v_{d0} from both sides to get:

$$\begin{aligned}
&\Delta v_d \\
&= -r\Delta i_d - L_q\left(\omega_0\Delta i_q + \Delta\omega\, i_{q0}\right) - kM_Q\left(\omega_0\Delta i_Q + i_{Q0}\Delta\omega\right) \\
&\quad - kM_G\left(\omega_0\Delta i_G + i_{G0}\Delta\omega\right) - L_d\Delta\dot{i}_d - kM_F\Delta\dot{i}_F - kM_D\Delta\dot{i}_D
\end{aligned} \tag{L-19}$$

Now combining terms in Δω, we obtain:

$$\begin{aligned}
&\Delta v_d \\
&= -r\Delta i_d - L_q\omega_0\Delta i_q - kM_Q\omega_0\Delta i_Q \\
&\quad - kM_G\omega_0\Delta i_G - L_d\Delta\dot{i}_d - kM_F\Delta\dot{i}_F - kM_D\Delta\dot{i}_D \\
&- \left(L_q i_{q0} + kM_Q i_{Q0} + kM_G i_{G0}\right)\Delta\omega
\end{aligned} \tag{L-20}$$

The last term in (L-20) is interesting. Let me show you why. We have previously derived that

$$
\begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \lambda_F \\ \lambda_D \\ \lambda_Q \\ \lambda_G \end{bmatrix} =
\begin{bmatrix}
L_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_d & 0 & \sqrt{\frac{3}{2}}M_F & \sqrt{\frac{3}{2}}M_D & 0 & 0 \\
0 & 0 & L_q & 0 & 0 & \sqrt{\frac{3}{2}}M_Q & \sqrt{\frac{3}{2}}M_G \\
0 & \sqrt{\frac{3}{2}}M_F & 0 & L_F & M_R & 0 & 0 \\
0 & \sqrt{\frac{3}{2}}M_D & 0 & M_R & L_D & 0 & 0 \\
0 & 0 & \sqrt{\frac{3}{2}}M_Q & 0 & 0 & L_Q & M_Y \\
0 & 0 & \sqrt{\frac{3}{2}}M_G & 0 & 0 & M_Y & L_G
\end{bmatrix}
\begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix}
$$

This is eq. 4.20 in text with the addition of the G-circuit. (We developed this equation in the notes on "mach_eqts." It was also used in the notes on per-unitization.)

From the above, we see that

$$\lambda_d = L_d i_d + kM_F i_F + kM_D i_D$$
$$\lambda_q = L_q i_q + kM_Q i_Q + kM_G i_G$$

Noting the equation for λ_q, we see that the coefficient in eq. (L-20) is λ_{q0}. Therefore, eq. (L-20) becomes:

$$
\begin{aligned}
&\Delta v_d \\
&= -r\Delta i_d - L_q\omega_0\Delta i_q - kM_Q\omega_0\Delta i_Q \\
&\quad - kM_G\omega_0\Delta i_G - L_d\Delta \dot{i}_d - kM_F\Delta \dot{i}_F - kM_D\Delta \dot{i}_D - \lambda_{q0}\Delta\omega
\end{aligned} \tag{L-21}
$$

And this is our linearized equation for the v_d voltage equation. Similar analysis applies for the other 7 state equations. Once this is done, we may form a matrix relation like that of eq. 6.20, pg. 211 in the text. Note, however, that we have one more state (the i_G state) than is represented in eq. 6.20, and the matrices will be 8×8 instead of 7×7, as shown in (L-22).

$$
\begin{bmatrix} \Delta v_d \\ -\Delta v_F \\ 0 \\ \Delta v_q \\ \Delta v_G \\ 0 \\ \Delta T_m \\ 0 \end{bmatrix}
= -\begin{bmatrix}
r & 0 & 0 & L_q\omega_0 & kM_G\omega_0 & kM_Q\omega_0 & \lambda_{q0} & 0 \\
0 & r_F & 0 & 0 & \text{—} & 0 & 0 & 0 \\
0 & 0 & r_D & 0 & \text{—} & 0 & 0 & 0 \\
-L_d\omega_0 & -kM_F\omega_0 & -kM_D\omega_0 & r & \text{—} & 0 & -\lambda_{d0} & 0 \\
\text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\
0 & 0 & 0 & 0 & \text{—} & r_Q & 0 & 0 \\
\frac{\lambda_{q0} - L_d i_{q0}}{3} & \frac{-kM_F i_{q0}}{3} & \frac{-kM_D i_{q0}}{3} & \frac{-\lambda_{d0} + L_q i_{d0}}{3} & \text{—} & \frac{kM_Q i_{d0}}{3} & -D & 0 \\
0 & 0 & 0 & 0 & \text{—} & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix} \Delta i_d \\ \Delta i_F \\ \Delta i_D \\ \Delta i_q \\ \Delta i_G \\ \Delta i_Q \\ \Delta\omega \\ \Delta\delta \end{bmatrix}
$$

$$
-\begin{bmatrix}
L_d & kM_F & kM_D & 0 & 0 & 0 & 0 & 0 \\
kM_F & L_F & M_R & 0 & \text{—} & 0 & 0 & 0 \\
kM_D & M_R & L_D & 0 & \text{—} & 0 & 0 & 0 \\
0 & 0 & 0 & L_q & \text{—} & kM_Q & 0 & 0 \\
\text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\
0 & 0 & 0 & kM_Q & \text{—} & L_Q & 0 & 0 \\
0 & 0 & 0 & 0 & \text{—} & 0 & -\tau_j & 0 \\
0 & 0 & 0 & 0 & \text{—} & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \Delta \dot{i}_d \\ \Delta \dot{i}_F \\ \Delta \dot{i}_D \\ \Delta \dot{i}_q \\ \Delta \dot{i}_G \\ \Delta \dot{i}_Q \\ \Delta\dot{\omega} \\ \Delta\dot{\delta} \end{bmatrix}
\tag{L-22}
$$

The resulting matrix relation (which includes the 6 voltage equations and the 2 inertial equations) can be written more compactly as:

$$\underline{v} = -\underline{K}\,\underline{x} - \underline{M}\,\underline{\dot{x}} \tag{eq. 6.21}$$

where the "Δ" on the variables v, x, and xdot is implied.

Then we may solve for $\underline{\dot{x}}$ according to:

$$\underline{\dot{x}} = -\underline{M}^{-1}\underline{K}\,\underline{x} - \underline{M}^{-1}\underline{v} \tag{eq. 6.22}$$

which is in the form of

$$\underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{v} \tag{eq. 6.23}$$

where

$$\underline{A} = -\underline{M}^{-1}\underline{K}$$

$$\underline{B} = -\underline{M}^{-1}$$

Section 6.3 shows, for a single machine connected to an infinite bus, in accounting for the loading through the vector v, the matrices M and K are both affected. Thus, the A-matrix is affected by loading conditions.

Final Exam Question #1.

a. Fill in the blanks in equation L-22 above.
b. Follow the procedure given in Section 6.3 to join the above model (after you fill in the blanks) with the linearized "load" equations of Equations (6.25). You should provide an answer similar to Equation (6.29) (notice this equation begins on page 213 and ends on page 214). Your matrices should be of dimension 8×8, with the additional dimension (relative to the corresponding equations given in the book) due to the presence of the G-winding.
c. Use the following data for this part of the problem:
   i. the pu data of obtained in Example 4.1 of the text;
   ii. the pu load data obtained in Example 5.2 of the text;
   iii. L_G=1.60 pu.
   iv. D=0.

   Develop the A-matrix as used in Equation (6.30) of the text and as expressed in Equation (6.31) of the text.
d. Find the eigenvalues of this system as done in Example 6.3 of the text. Compare your answer to the eigenvalues obtained in Example 6.3. How are they different and why are they different?

You should bring your work to the final exam and hand it in with the rest of your exam material.