

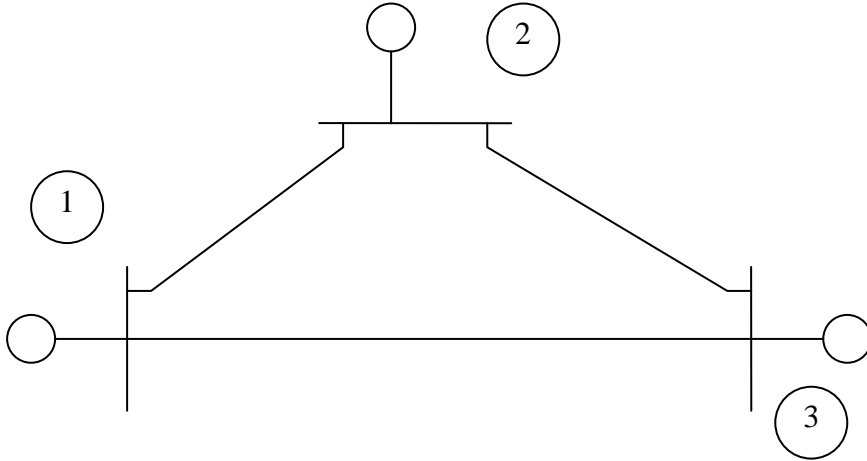
State-Space Model for a Multi-Machine System

These notes parallel section 3.4 in the text. Note that we are dealing with classically modeled machines (IEEE Type 0.0), constant impedance loads, and a network reduced to its internal machine terminals.

We have found that the linearized swing equation is given by:

$$\frac{2H_i}{\omega_{\text{Re}}} \frac{d^2 \Delta \delta_i}{dt^2} + \sum_{\substack{i=1 \\ i \neq j}}^n P_{Sij} \Delta \delta_{ij} = 0, \quad i=1, \dots, n \quad (\text{eq. 3.26})$$

where it is important to note that we have assumed no damping. Let's consider writing the linearized swing equations for a test system (see example 3.1 in text), as shown below.



The three swing equations are:

$$\begin{aligned} \frac{2H_1}{\omega_{\text{Re}}} \frac{d^2 \Delta \delta_1}{dt^2} + P_{S12} \Delta \delta_{12} + P_{S13} \Delta \delta_{13} &= 0 \\ \frac{2H_2}{\omega_{\text{Re}}} \frac{d^2 \Delta \delta_2}{dt^2} + P_{S21} \Delta \delta_{21} + P_{S23} \Delta \delta_{23} &= 0 \\ \frac{2H_3}{\omega_{\text{Re}}} \frac{d^2 \Delta \delta_3}{dt^2} + P_{S31} \Delta \delta_{31} + P_{S32} \Delta \delta_{32} &= 0 \end{aligned}$$

One important fact: The stability of a power system depends on relative rotor angles $\Delta\delta_{ij}$ NOT absolute rotor angles $\Delta\delta_i$. This is because *synchronism* is a relative phenomenon. That is, it makes no sense to say “generator 1 is in synchronism.” Rather, we must say with what it is in synchronism, i.e., “Generator 1 is in synchronism with generators 2 and 3,” or “Generator 1 is in synchronism with the rest of the system.”

So we need to define our states in terms of relative rotor angles. In the above equations, the states (derivatives) are in terms of absolute rotor angle. We can deal with this in the following way.

First, multiply through each equation by $\omega_{Re}/2H_i$, resulting in:

$$\begin{aligned}\frac{d^2\Delta\delta_1}{dt^2} + \frac{\omega_{Re}}{2H_1} [P_{S12}\Delta\delta_{12} + P_{S13}\Delta\delta_{13}] &= 0 \\ \frac{d^2\Delta\delta_2}{dt^2} + \frac{\omega_{Re}}{2H_2} [P_{S21}\Delta\delta_{21} + P_{S23}\Delta\delta_{23}] &= 0 \\ \frac{d^2\Delta\delta_3}{dt^2} + \frac{\omega_{Re}}{2H_3} [P_{S31}\Delta\delta_{31} + P_{S32}\Delta\delta_{32}] &= 0\end{aligned}$$

Now subtract the last equation from each of the other two. When we do this, the derivative terms will be affected in the following way (and this is the main motivation for making this subtraction):

$$\frac{d^2\Delta\delta_1}{dt^2} - \frac{d^2\Delta\delta_3}{dt^2} = \frac{d^2(\Delta\delta_1 - \Delta\delta_3)}{dt^2} = \frac{d^2\Delta\delta_{13}}{dt^2}$$

which is $d\omega_{13}/dt$. Note, however, that this very convenient substitution of variable will not help if damping is modeled ($D_i\omega_i$) in the swing equation AND the damping is nonuniform (nonuniform damping means that $D_i \neq D_k$ for $i \neq k$) because then it is not possible to combine the corresponding speed variables. Let's look at this issue by re-writing the above equations with nonuniform damping.

$$\frac{d^2\Delta\delta_1}{dt^2} + \frac{D_1}{2H_1} \frac{d\Delta\delta_1}{dt} + \frac{\omega_{\text{Re}}}{2H_1} [P_{S12}\Delta\delta_{12} + P_{S13}\Delta\delta_{13}] = 0$$

$$\frac{d^2\Delta\delta_2}{dt^2} + \frac{D_2}{2H_2} \frac{d\Delta\delta_2}{dt} + \frac{\omega_{\text{Re}}}{2H_2} [P_{S21}\Delta\delta_{21} + P_{S23}\Delta\delta_{23}] = 0$$

$$\frac{d^2\Delta\delta_3}{dt^2} + \frac{D_3}{2H_3} \frac{d\Delta\delta_3}{dt} + \frac{\omega_{\text{Re}}}{2H_3} [P_{S31}\Delta\delta_{31} + P_{S32}\Delta\delta_{32}] = 0$$

Subtracting the last equation from the other two (in the case of the first equation) affects the derivative terms in the following way:

$$\frac{d^2\Delta\delta_1}{dt^2} - \frac{d^2\Delta\delta_3}{dt^2} = \frac{d^2(\Delta\delta_1 - \Delta\delta_3)}{dt^2} = \frac{d^2\Delta\delta_{13}}{dt^2} \text{ (as before)}$$

$$\frac{D_1}{2H_1} \frac{d\Delta\delta_1}{dt} - \frac{D_3}{2H_3} \frac{d\Delta\delta_3}{dt} \rightarrow \text{Cannot do more with this!}$$

It will work if the damping is uniform, however, implying that the ratio D_i/H_i is the same for all i . In this case, we get, for example,

$$\frac{D_1}{2H_1} \frac{d\Delta\delta_1}{dt} - \frac{D_3}{2H_3} \frac{d\Delta\delta_3}{dt} = \frac{D_1}{2H_1} \left(\frac{d\Delta\delta_1}{dt} - \frac{d\Delta\delta_3}{dt} \right) = \frac{D_1}{2H_1} \frac{d\Delta\delta_{13}}{dt} = \frac{D_1}{2H_1} \Delta\omega_{13}$$

The implication is that we can ALWAYS reduce the number of states by 1 due to the ability to use relative angles. But an *additional* reduction of states by 1 due to the ability to use relative speeds only occurs in the cases of *no damping* or of *uniform damping*. In general, we do need to model nonuniform damping and therefore we are usually only able to get the state reduction for relative angles, but not for relative speeds.

In this particular special case of no damping, we get:

$$\frac{d^2 \Delta \delta_{13}}{dt^2} + \frac{\omega_{\text{Re}}}{2H_1} [P_{S12} \Delta \delta_{12} + P_{S13} \Delta \delta_{13}] - \frac{\omega_{\text{Re}}}{2H_3} [P_{S31} \Delta \delta_{31} + P_{S32} \Delta \delta_{32}] = 0$$

$$\frac{d^2 \Delta \delta_{23}}{dt^2} + \frac{\omega_{\text{Re}}}{2H_2} [P_{S21} \Delta \delta_{21} + P_{S23} \Delta \delta_{23}] - \frac{\omega_{\text{Re}}}{2H_3} [P_{S31} \Delta \delta_{31} + P_{S32} \Delta \delta_{32}] = 0$$

Recognizing that $\Delta \delta_{31} = -\Delta \delta_{13}$ and that $\Delta \delta_{32} = -\Delta \delta_{23}$, we may change the sign of the second term in each equation if we also make this change of variables. This results in:

$$\frac{d^2 \Delta \delta_{13}}{dt^2} + \frac{\omega_{\text{Re}}}{2H_1} [P_{S12} \Delta \delta_{12} + P_{S13} \Delta \delta_{13}] + \frac{\omega_{\text{Re}}}{2H_3} [P_{S31} \Delta \delta_{13} + P_{S32} \Delta \delta_{23}] = 0$$

$$\frac{d^2 \Delta \delta_{23}}{dt^2} + \frac{\omega_{\text{Re}}}{2H_2} [P_{S21} \Delta \delta_{21} + P_{S23} \Delta \delta_{23}] + \frac{\omega_{\text{Re}}}{2H_3} [P_{S31} \Delta \delta_{13} + P_{S32} \Delta \delta_{23}] = 0$$

eq. (#)

So we have states are $\Delta \delta_{13}$ and $\Delta \delta_{23}$. But note carefully that there are some other variables, namely $\Delta \delta_{12}$, $\Delta \delta_{21}$.

This means we have 4 variables and only 2 equations.

Can we express $\Delta \delta_{12}$ and $\Delta \delta_{21}$ in terms of $\Delta \delta_{13}$ and $\Delta \delta_{23}$. Clearly, since $\Delta \delta_{12} = -\Delta \delta_{21}$, if we can do it for one, we can do it for the other.

This is done by noting first that

$$\Delta \delta_{12} + \Delta \delta_{23} + \Delta \delta_{31} = 0 \quad (\text{eq. *})$$

We can prove this as follows:

$$\begin{aligned} & \Delta \delta_{12} + \Delta \delta_{23} + \Delta \delta_{31} \\ &= \Delta \delta_1 - \Delta \delta_2 + \Delta \delta_2 - \Delta \delta_3 + \Delta \delta_3 - \Delta \delta_1 = 0 \end{aligned}$$

Therefore, from eq. (*), we can write that

$$\Delta \delta_{12} = -\Delta \delta_{23} - \Delta \delta_{31}$$

Reversing the subscript order of each term on the right-hand-side, and changing signs, we get:

$$\Delta\delta_{12} = -\Delta\delta_{23} + \Delta\delta_{13} \quad (\text{eq. **})$$

Then, since $\Delta\delta_{12} = -\Delta\delta_{21}$, we get

$$\Delta\delta_{21} = \Delta\delta_{23} - \Delta\delta_{13} \quad (\text{eq. ***)}$$

Substituting eq. (**) and (***) into eq (#), we obtain

$$\begin{aligned} \frac{d^2\Delta\delta_{13}}{dt^2} + \frac{\omega_{\text{Re}}}{2H_1} [P_{S12}\Delta\delta_{13} - P_{S12}\Delta\delta_{23} + P_{S13}\Delta\delta_{13}] + \frac{\omega_{\text{Re}}}{2H_3} [P_{S31}\Delta\delta_{13} + P_{S32}\Delta\delta_{23}] &= 0 \\ \frac{d^2\Delta\delta_{23}}{dt^2} + \frac{\omega_{\text{Re}}}{2H_2} [P_{S21}\Delta\delta_{23} - P_{S21}\Delta\delta_{13} + P_{S23}\Delta\delta_{23}] + \frac{\omega_{\text{Re}}}{2H_3} [P_{S31}\Delta\delta_{13} + P_{S32}\Delta\delta_{23}] &= 0 \end{aligned}$$

Gathering terms in each variable, we get two differential equations:

$$\begin{aligned} \frac{d^2\Delta\delta_{13}}{dt^2} + \underbrace{\left[\frac{\omega_{\text{Re}}}{2H_1} P_{S12} + \frac{\omega_{\text{Re}}}{2H_1} P_{S13} + \frac{\omega_{\text{Re}}}{2H_3} P_{S31} \right]}_{\alpha_{11}} \Delta\delta_{13} + \underbrace{\left[\frac{\omega_{\text{Re}}}{2H_3} P_{S32} - \frac{\omega_{\text{Re}}}{2H_1} P_{S12} \right]}_{\alpha_{12}} \Delta\delta_{23} &= 0 \\ \frac{d^2\Delta\delta_{23}}{dt^2} + \underbrace{\left[\frac{\omega_{\text{Re}}}{2H_3} P_{S31} - \frac{\omega_{\text{Re}}}{2H_2} P_{S21} \right]}_{\alpha_{21}} \Delta\delta_{13} + \underbrace{\left[\frac{\omega_{\text{Re}}}{2H_2} P_{S21} + \frac{\omega_{\text{Re}}}{2H_2} P_{S23} + \frac{\omega_{\text{Re}}}{2H_3} P_{S32} \right]}_{\alpha_{22}} \Delta\delta_{23} &= 0 \end{aligned}$$

Denote the coefficients of the above differential equations as α_{11} , α_{12} , α_{21} , and α_{22} , where (assuming the last equation, for bus n, is the one that gets subtracted off in the above steps):

$$\alpha_{ii} = \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\omega_{\text{Re}}}{2H_i} P_{Sij} \right\} + \frac{\omega_{\text{Re}}}{2H_n} P_{Sni} \quad \alpha_{ij} = \frac{\omega_{\text{Re}}}{2H_n} P_{Snj} - \frac{\omega_{\text{Re}}}{2H_i} P_{Sij}$$

Note these expressions differ from those given in the text's addendum, pg. 650, because my alpha's here are defined on the left-hand-side of the equation, whereas the text's equations are defined on the right-hand-side of the equation.

Using these alphas, I rewrite the differential equations as

$$\frac{d^2 \Delta \delta_{13}}{dt^2} + \alpha_{11} \Delta \delta_{13} + \alpha_{12} \Delta \delta_{23} = 0$$

$$\frac{d^2 \Delta \delta_{23}}{dt^2} + \alpha_{21} \Delta \delta_{13} + \alpha_{22} \Delta \delta_{23} = 0$$

We can now convert these second order linear differential equations into first order linear differential equations, in order to develop a state-space form.

We do this by recognizing that

$$\Delta \omega_{13} = \frac{d\delta_{13}}{dt}, \quad \Delta \omega_{23} = \frac{d\delta_{23}}{dt}$$

Then, the above two second order differential equations become four first order differential equations, as follows:

$$\Delta \dot{\omega}_{13} = -\alpha_{11} \Delta \delta_{13} - \alpha_{12} \Delta \delta_{23} = 0$$

$$\Delta \dot{\omega}_{23} = -\alpha_{21} \Delta \delta_{13} - \alpha_{22} \Delta \delta_{23} = 0$$

$$\Delta \dot{\delta}_{13} = \Delta \omega_{13}$$

$$\Delta \dot{\delta}_{23} = \Delta \omega_{23}$$

So let's define the state vector as

$$\underline{x} = \begin{bmatrix} \Delta \delta_{13} \\ \Delta \delta_{23} \\ \Delta \omega_{13} \\ \Delta \omega_{23} \end{bmatrix}$$

Note for our 3 machine system, we have only 4 states due to the state reduction for relative angles and relative speeds.

Then

$$\dot{\underline{x}} = A \underline{x}$$

More explicitly,

$$\begin{bmatrix} \Delta \dot{\delta}_{13} \\ \Delta \dot{\delta}_{23} \\ \Delta \dot{\omega}_{13} \\ \Delta \dot{\omega}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_{11} & -\alpha_{12} & 0 & 0 \\ -\alpha_{21} & -\alpha_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta_{13} \\ \Delta \delta_{23} \\ \Delta \omega_{13} \\ \Delta \omega_{23} \end{bmatrix}$$

Your text on page 63 shows the computation of the alpha-coefficients for the 9-bus, 3-generator system of Fig. 2.19. It is shown that

$$\alpha_{11}=104.096$$

$$\alpha_{12}=59.524$$

$$\alpha_{21}=33.841$$

$$\alpha_{22}=153.460$$

Then, the state-space equation is:

$$\begin{bmatrix} \Delta \dot{\delta}_{13} \\ \Delta \dot{\delta}_{23} \\ \Delta \dot{\omega}_{13} \\ \Delta \dot{\omega}_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -104.096 & -59.524 & 0 & 0 \\ -33.841 & -153.460 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta_{13} \\ \Delta \delta_{23} \\ \Delta \omega_{13} \\ \Delta \omega_{23} \end{bmatrix}$$

Question is, now, what to do with the above in order to obtain useful information about the small-disturbance behavior of our system. We will investigate this next....