Load Equations

Throughout all of chapter 4, our focus is on the machine itself, therefore we will only perform a very simple treatment of the network in order to see a complete model.

So let's look at a single machine connected to an infinite bus, as illustrated in Fig. 1 below.

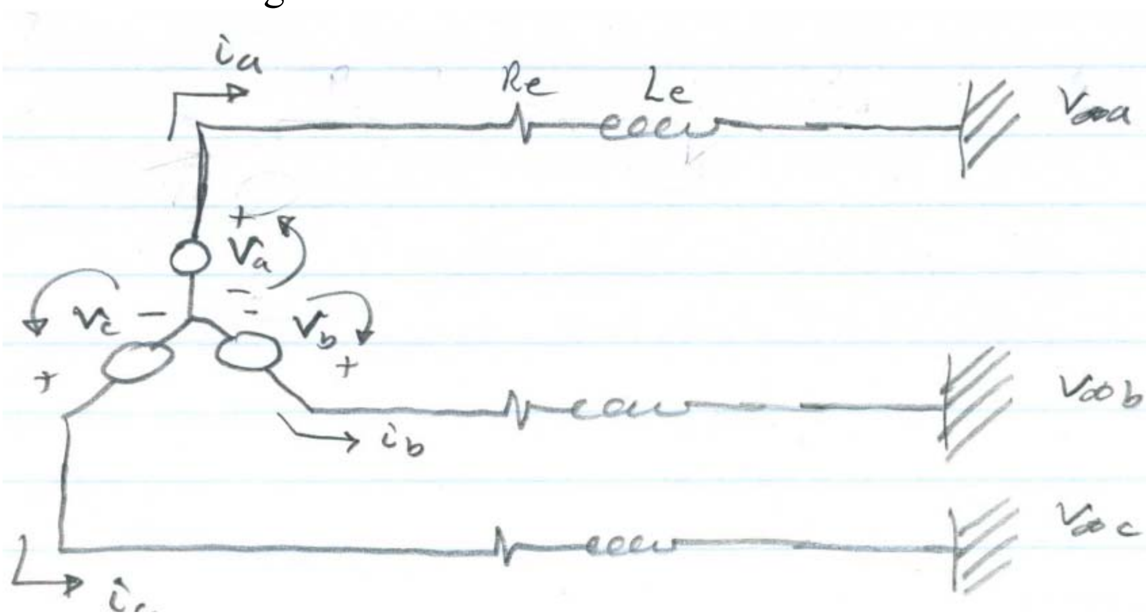

Fig. 1

From KVL, we have

$$\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} v_{\infty,a} \\ v_{\infty,b} \\ v_{\infty,c} \end{bmatrix} + R_e \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{U}} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + L_e \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{U}} \begin{bmatrix} \dot{i}_a \\ \dot{i}_b \\ \dot{i}_c \end{bmatrix}$$

$$\Rightarrow \underline{v}_{abc} = \underline{v}_{\infty,abc} + R_e \underline{U}\,\underline{i}_{abc} + L_e \underline{U}\,\dot{\underline{i}}_{abc}$$

Now use Park's transformation to obtain:

$$\underline{v}_{0dq} = \underline{P}\underline{v}_{abc} = \underline{P}\underline{v}_{\infty,abc} + R_e \underbrace{\underline{P}\underline{U}}_{\underline{P}} \underline{i}_{abc} + L_e \underbrace{\underline{P}\underline{U}}_{\underline{P}} \underline{\dot{i}}_{abc}$$

$$\Rightarrow \underline{v}_{0dq} = \underbrace{\underline{v}_{\infty,0dq}}_{TERM1} + \underbrace{R_e \underline{i}_{0dq}}_{TERM2} + \underbrace{L_e \underline{P}\underline{\dot{i}}_{abc}}_{TERM3} \qquad (1)$$

We would like to express v_d and v_q as a function of state variables (the 0dq currents for the current model or the 0dq flux linkages for the flux linkage model). Let's consider each term.

TERM1:

$$\underline{v}_{\infty,0dq} = \underline{P}\underline{v}_{\infty,abc}$$

So what is $\underline{v}_{\infty,abc}$?

A good assumption for purposes of stability assessment is that they are a set of balanced voltages having rms value of V_∞, i.e.,

$$\underline{v}_{\infty,abc} = \begin{bmatrix} v_{\infty,a} \\ v_{\infty,b} \\ v_{\infty,c} \end{bmatrix} = \begin{bmatrix} \sqrt{2}V_\infty \cos(\omega_{\mathrm{Re}} t + \alpha) \\ \sqrt{2}V_\infty \cos(\omega_{\mathrm{Re}} t + \alpha - 120) \\ \sqrt{2}V_\infty \cos(\omega_{\mathrm{Re}} t + \alpha + 120) \end{bmatrix}$$

Hit the above with Park's transformation matrix to obtain:

$$\underline{v}_{\infty,0dq} = \underline{P}\underline{v}_{\infty,abc} = \sqrt{3}V_\infty \begin{bmatrix} 0 \\ -\sin(\delta - \alpha) \\ \cos(\delta - \alpha) \end{bmatrix}$$

And so we see that we can the balanced AC voltages transform to a set of DC voltages, as we have observed before.

TERM2: This one is easy as it is already written in terms of the 0dq currents.

TERM3: We must be a little careful here. It is tempting to use $\underline{\dot{i}}_{0dq} = \underline{P}\underline{\dot{i}}_{abc}$. But is this true?

Let's back up and recall that

$$\underline{i}_{0dq} = \underline{P}\underline{i}_{abc}$$

Taking the derivative of the left-hand-side, we obtain:

$\dot{\underline{i}}_{0dq} = \underline{P}\dot{\underline{i}}_{abc} + \dot{\underline{P}}\underline{i}_{abc}$ (2)

And this proves that $\dot{\underline{i}}_{0dq} \neq \underline{P}\dot{\underline{i}}_{abc}$.

But we know that $\underline{i}_{abc} = \underline{P}^{-1}\underline{i}_{0dq}$, and using this in (2) results in

$\dot{\underline{i}}_{0dq} = \underline{P}\dot{\underline{i}}_{abc} + \dot{\underline{P}}\underline{P}^{-1}\underline{i}_{0dq}$

Isolating the first term on the right results in

$$\underline{P}\dot{\underline{i}}_{abc} = \dot{\underline{i}}_{0dq} - \dot{\underline{P}}\underline{P}^{-1}\underline{i}_{0dq}$$

Recalling that term3 is $L_e\underline{P}\dot{\underline{i}}_{abc}$, we multiple the above by L_e to obtain term3:

$L_e\underline{P}\dot{\underline{i}}_{abc} = L_e\left(\dot{\underline{i}}_{0dq} - \dot{\underline{P}}\underline{P}^{-1}\underline{i}_{0dq}\right)$

You may recall now that in Section 4.4 (notes on “macheqts”, pp. 22-23) that we found

$$\dot{\underline{P}}\underline{P}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

So term3 becomes

$$L_e\underline{P}\dot{\underline{i}}_{abc} = L_e\left(\dot{\underline{i}}_{0dq} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}\underline{i}_{0dq}\right)$$

Or

$$L_e\underline{P}\dot{\underline{i}}_{abc} = L_e\left(\underline{\begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix}} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}\underline{\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix}}\right) = L_e\left(\underline{\begin{bmatrix} \dot{i}_0 \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix}} - \omega\underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}}\right) = L_e\left(\dot{\underline{i}}_{0dq} - \omega\underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}}\right)$$

Substitution of our terms 1, 2, and 3 back into eq. (1) results in

$$\underline{v}_{0dq} = \sqrt{3}V_{\infty}\begin{bmatrix} 0 \\ -\sin(\delta-\alpha) \\ \cos(\delta-\alpha) \end{bmatrix} + R_e \underline{i}_{0dq} + L_e \dot{\underline{i}}_{0dq} - L_e \omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \qquad (4.149)$$

Now we need to incorporate this into our state-space model.

We have three different models.

A. Current state-space model

B. Flux-linkage-state-space model with λ_{AD} and λ_{AQ} (for modeling saturation)

C. Flux-linkage-state-space model with λ_{AD} and λ_{AQ} eliminated (and so without the ability to modeling saturation)

I have hand-written notes where I went through the details of this for models (A) and (C), although I did not include the G-winding. I want to do that but have just not had time to do it. And so I simply provide the results for the model without the G-winding.

<u>A. Current state-space model (See section 4.13.2)</u>

Recall that the current state-space model is

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \dot{i}_G \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} \multicolumn{1}{c}{-\underline{L}^{-1}(\underline{R}+\omega\underline{N})} & \underline{0} \\ \begin{bmatrix} \frac{-L_d i_q}{3\tau_j} & \frac{-kM_F i_q}{3\tau_j} & \frac{-kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & \frac{kM_G i_d}{3\tau_j} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{-D}{\tau_j} & 0 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \\ \omega \\ \delta \end{bmatrix} + \begin{bmatrix} \underline{-\underline{L}^{-1}\underline{v}} \\ \begin{bmatrix} \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \end{bmatrix}$$

(4.103')

where the submatrices are given by

$$\underline{R} = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r_F & 0 & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & r_G \end{bmatrix}; \qquad \underline{N} = \begin{bmatrix} 0 & 0 & 0 & L_q & kM_Q & kM_G \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -L_d & -kM_F & -kM_D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{L} = \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_Q & kM_G \\ 0 & 0 & 0 & kM_Q & L_Q & M_Y \\ 0 & 0 & 0 & kM_G & M_Y & L_G \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} v_d \\ -v_F \\ 0 \\ v_q \\ 0 \\ 0 \end{bmatrix}; \qquad \underline{i} = \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix}$$

Incorporating into our load equations, eq. (4.149), into our state-space current model, (4.103'), results in

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \left[\begin{array}{ccccc|cc} \multicolumn{5}{c|}{-\hat{\mathbf{L}}^{-1}(\hat{\mathbf{R}} + \omega\hat{\mathbf{N}})} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline -\frac{L_d i_q}{3\tau_j} & -\frac{kM_F i_q}{3\tau_j} & -\frac{kM_D i_q}{3\tau_j} & \frac{L_q i_d}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & -\frac{D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ \omega \\ \delta \end{bmatrix} + \left[\begin{array}{c|cc} -\hat{\mathbf{L}}^{-1} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & 1 & 0 \\ & 0 & 1 \end{array}\right] \begin{bmatrix} -K\sin\gamma \\ -v_F \\ 0 \\ K\cos\gamma \\ 0 \\ \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \tag{4.154}$$

where the matrices with the hats above them, i.e., $\hat{L}, \hat{R}, \hat{N}$, are exactly as the unhat-ed versions above, except that

- Wherever you see r, replace it with $r + R_e$
- Wherever you see L_d, replace it with $L_d + L_e$
- Wherever you see L_q, replace it with $L_q + L_e$

Note that:
K=√3 V_∞ (**not** the same K as used in the saturation notes), and γ=δ-α.

Your text makes a useful remark (pg. 117) in saying that,

"The system described by (4.154) is now in the form of …
$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$, where $\underline{x}^T = [i_d \quad i_F \quad i_D \quad i_q \quad i_Q \quad \omega \quad \delta]$."

(and, of course, i_G)…

"The function f is a nonlinear function of the state variables and t, and u contains the system driving functions, which are v_F and T_m. The loading effect of the transmission line is incorporated in the matrices $\hat{L}, \hat{R}, \hat{N}$. The infinite bus voltage V_∞ appears in the terms Ksinγ and Kcosγ. Note also that these latter terms are not driving functions, but rather nonlinear functions of the state variable δ."

C. Flux-linkage-state-space model with λ_{AD}, λ_{AQ} eliminated (so without ability to modeling saturation) (See section 4.13.3 of text).

Recall the state-space model of eq. (4.138)

$$
\begin{bmatrix} \dot\lambda_d \\ \dot\lambda_F \\ \dot\lambda_D \\ \dot\lambda_q \\ \dot\lambda_Q \\ \dot\omega \\ \dot\delta \end{bmatrix}
=
\begin{bmatrix}
-\frac{r}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & \frac{r}{\ell_d}\frac{L_{MD}}{\ell_F} & \frac{r}{\ell_d}\frac{L_{MD}}{\ell_D} & -\omega & 0 & 0 & 0 \\
\frac{r_F}{\ell_F}\frac{L_{MD}}{\ell_d} & -\frac{r_F}{\ell_F}\left(1-\frac{L_{MD}}{\ell_F}\right) & \frac{r_F}{\ell_F}\frac{L_{MD}}{\ell_D} & 0 & 0 & 0 & 0 \\
\frac{r_D}{\ell_D}\frac{L_{MD}}{\ell_d} & \frac{r_D}{\ell_D}\frac{L_{MD}}{\ell_F} & -\frac{r_D}{\ell_D}\left(1-\frac{L_{MD}}{\ell_D}\right) & 0 & 0 & 0 & 0 \\
\omega & 0 & 0 & -\frac{r}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & \frac{r}{\ell_q}\frac{L_{MQ}}{\ell_Q} & 0 & 0 \\
0 & 0 & 0 & \frac{r_Q}{\ell_Q}\frac{L_{MQ}}{\ell_q} & -\frac{r_Q}{\ell_Q}\left(1-\frac{L_{MQ}}{\ell_Q}\right) & 0 & 0 \\
-\frac{L_{MD}}{3\tau_j\ell_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_F}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_D}\lambda_q & \frac{L_{MQ}}{3\tau_j\ell_q^2}\lambda_d & \frac{L_{MQ}}{3\tau_j\ell_q\ell_Q}\lambda_d & -\frac{D}{\tau_j} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \lambda_q \\ \lambda_Q \\ \omega \\ \delta \end{bmatrix}
+
\begin{bmatrix} -v_d \\ v_F \\ 0 \\ -v_q \\ 0 \\ \frac{T_m}{\tau_j} \\ -1 \end{bmatrix}
\tag{4.138}
$$

We see we need to incorporate the load equations, (4.149), through the v_d, v_q terms.These equations are repeated here for convenience:

$$
\underline{v}_{0dq} = \sqrt{3}V_\infty \begin{bmatrix} 0 \\ -\sin(\delta-\alpha) \\ \cos(\delta-\alpha) \end{bmatrix} + R_e \underline{i}_{0dq} + L_e \dot{\underline{i}}_{0dq} - L_e\omega \underline{\begin{bmatrix} 0 \\ -i_q \\ i_d \end{bmatrix}} \qquad (4.149)
$$

However, this time we need the load equations in terms of flux linkages. This takes some work, which I have done in detailed hand-written notes (will be happy to provide if you want them). This results in eqts. 4.57, 4.58 in your text, repeated here.

$$
\begin{aligned}
&\left[1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right]\dot\lambda_d - \frac{L_e L_{MD}}{\ell_d\ell_F}\dot\lambda_F - \frac{L_e L_{MD}}{\ell_d\ell_D}\dot\lambda_D = \frac{-\hat R}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\lambda_d + \frac{\hat R L_{MD}}{\ell_d\ell_F}\lambda_F \\
&\quad + \frac{\hat R L_{MD}}{\ell_d\ell_D}\lambda_D - \omega\left[1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right]\lambda_q + \frac{\omega L_e L_{MQ}}{\ell_q\ell_Q}\lambda_Q + \sqrt{3}V_\infty\sin(\delta-\alpha)
\end{aligned}
\tag{4.157}
$$

Similarly, we combine (4.156) with (4.136) to get

$$
\begin{aligned}
&\left[1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right]\dot\lambda_q - \frac{L_e L_{MQ}}{\ell_q\ell_Q}\dot\lambda_Q = -\frac{\hat R}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\lambda_q + \frac{\hat R L_{MQ}}{\ell_q\ell_Q}\lambda_Q \\
&\quad + \omega\left[1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right]\lambda_d - \frac{\omega L_e L_{MD}}{\ell_d\ell_F}\lambda_F - \frac{\omega L_e L_{MD}}{\ell_d\ell_D}\lambda_D - \sqrt{3}V_\infty\cos(\delta-\alpha)
\end{aligned}
\tag{4.158}
$$

Note in these two equations that there are several derivative terms and so we cannot "cleanly" use these equations to simply replace the derivatives on λ_d and λ_q in the flux-linkage state-space model (we were able to do so with the current state-space model).

Rather, we have to create a pre-multiplier matrix $\underline{T}$ such that

$$\underline{T}\,\dot{\underline{x}} = \underline{C}\,\underline{x} + \underline{D}$$

where

$$\underline{x} = \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \lambda_q \\ \lambda_Q \\ \omega \\ \delta \end{bmatrix}$$

Note we need to include λ_G here, and then augment the $\underline{T}$, $\underline{C}$, and $\underline{D}$ matrices accordingly.

And $\underline{T}$, $\underline{C}$, and $\underline{D}$ are given by

$$\mathbf{T} = \left[\begin{array}{ccc|cc|cc} 1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & -\frac{L_e L_{MD}}{\ell_d \ell_F} & -\frac{L_e L_{MD}}{\ell_d \ell_D} & & & & \\ 0 & 1 & 0 & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c}{\mathbf{0}} \\ 0 & 0 & 1 & & & & \\ \hline & \mathbf{0} & & 1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & -\frac{L_e L_{MQ}}{\ell_q \ell_Q} & \multicolumn{2}{c}{\mathbf{0}} \\ & & & 0 & 1 & & \\ \hline & \mathbf{0} & & \multicolumn{2}{c|}{\mathbf{0}} & 1 & 0 \\ & & & & & 0 & 1 \end{array}\right] \qquad (4.160)$$

$$
\mathbf{C} = \left[\begin{array}{ccc:cc:cc}
-\frac{\hat{R}}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right) & \frac{\hat{R}L_{MD}}{\ell_d\ell_F} & \frac{\hat{R}L_{MD}}{\ell_d\ell_D} & -\omega\left[1+\frac{L_e}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right)\right] & \frac{\omega L_e L_{MQ}}{\ell_q\ell_Q} & 0 & 0 \\
\frac{r_F L_{MD}}{\ell_F\ell_d} & -\frac{r_F}{\ell_F}\left(1-\frac{L_{MD}}{\ell_F}\right) & \frac{r_F L_{MD}}{\ell_F\ell_D} & 0 & 0 & 0 & 0 \\
\frac{r_D L_{MD}}{\ell_D\ell_d} & \frac{r_D L_{MD}}{\ell_D\ell_F} & -\frac{r_D}{\ell_D}\left(1-\frac{L_{MD}}{\ell_D}\right) & 0 & 0 & 0 & 0 \\
\hdashline
\omega\left[1+\frac{L_e}{\ell_d}\left(1-\frac{L_{MD}}{\ell_d}\right)\right] & -\frac{\omega L_e L_{MD}}{\ell_d\ell_F} & -\frac{\omega L_e L_{MD}}{\ell_d\ell_D} & -\frac{\hat{R}}{\ell_q}\left(1-\frac{L_{MQ}}{\ell_q}\right) & \frac{\hat{R}L_{MQ}}{\ell_q\ell_Q} & 0 & 0 \\
0 & 0 & 0 & \frac{r_Q L_{MQ}}{\ell_Q\ell_q} & -\frac{r_Q}{\ell_Q}\left(1-\frac{L_{MQ}}{\ell_Q}\right) & 0 & 0 \\
\hdashline
-\frac{L_{MD}}{3\tau_j\ell_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j\ell_d\ell_F}\lambda_q & \frac{L_{MD}}{3\tau_j\ell_d\ell_D}\lambda_q & \frac{L_{MQ}}{3\tau_j\ell_d^2}\lambda_d & \frac{L_{MQ}}{3\tau_j\ell_q\ell_Q}\lambda_d & -\frac{D}{\tau_j} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] \quad (4.161)
$$

$$
\mathbf{D} = \begin{bmatrix} \sqrt{3}V_\infty \sin(\delta-\alpha) \\ v_F \\ 0 \\ -\sqrt{3}V_\infty \cos(\delta-\alpha) \\ 0 \\ T_m/\tau_j \\ -1 \end{bmatrix} \quad (4.162)
$$

Then we can pre-multiple both sides by $\underline{T}^{-1}$ to obtain

$$\dot{\underline{x}} = \underline{T}^{-1}\underline{C}\,\underline{x} + \underline{T}^{-1}\underline{D} \quad (4.163)$$

Equation (4.163) describes the complete system of interest to us at this point, i.e., the system of Fig. 1 at the beginning of these notes. To use it, we need the initial states $\underline{x}(0)$ which are found by solving $\underline{T}\,\dot{\underline{x}} = \underline{C}\,\underline{x} + \underline{D} = \underline{0}$, via $\underline{x} = \underline{C}^{-1}\underline{D}$ where vector $\underline{D}$ provides system loading information.

Then, if we perturb the system by setting, for example, $V_\infty=0$ for a few cycles, then the response can be obtained by solving eq. (4.163) using numerical integration.