Linearization of the Swing Equation

We will cover sections 2.5.2-2.6 and beginning of Section 3.3 in these notes.

1.0 Single machine-infinite bus case

Consider a single machine connected to an infinite bus, as shown in Fig. 1 below.

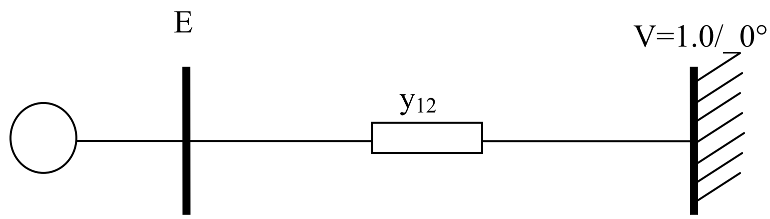

Fig. 1

The admittance matrix is given by

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} y_{12} & -y_{12} \\ -y_{12} & y_{12} \end{bmatrix} \tag{1}$$

Let's assume the machine is modeled by the swing equation with damping, given by

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d^2\delta}{dt^2} + \frac{D}{\omega_{\mathrm{Re}}}\frac{d\delta}{dt} = P_m - P_e = P_m - P_M \sin(\delta - \gamma) \tag{2}$$

where

- $P_M = |E||V||Y_{12}|$
- $Y_{12} = |Y_{12}| \angle \theta_{12}$
- $\gamma = \theta_{12} - \pi/2$ (enables us to use sin instead of cos – see pg. 27 of text)

Now let the angle δ change by a small amount. Then

$$\delta = \delta_0 + \Delta\delta \Rightarrow \frac{d\delta}{dt} = \frac{d\Delta\delta}{dt}, \quad \frac{d^2\delta}{dt^2} = \frac{d^2\Delta\delta}{dt^2} \tag{3}$$

Also recall that by Taylor series,

$$\sin x = \sin(x_0 + \Delta x) = \sin x_0 + \left.\frac{d \sin x}{dx}\right|_{x_0} \Delta x = \sin x_0 + (\cos x_0)\Delta x \tag{4}$$

Then we also see that

$$\sin(\delta - \gamma) = \sin(\delta_0 - \gamma + \Delta\delta) = \sin(\delta_0 - \gamma) + (\cos(\delta_0 - \gamma))\Delta\delta \tag{5}$$

(Eqt. 3.3)

Applying (3) to the left-hand-side of (2) and (5) to the right-hand-side of (2), we obtain

$$\begin{aligned} \frac{2H}{\omega_{\text{Re}}} \frac{d^2 \Delta\delta}{dt^2} + \frac{D}{\omega_{\text{Re}}} \frac{d\Delta\delta}{dt} &= P_m - P_M \sin(\delta_0 - \gamma + \Delta\delta) \\ &= P_m - P_M [\sin(\delta_0 - \gamma) + (\cos(\delta_0 - \gamma))\Delta\delta] \\ &= P_m - P_M \sin(\delta_0 - \gamma) - P_M (\cos(\delta_0 - \gamma))\Delta\delta] \end{aligned} \tag{6}$$

But

$$P_m = P_M \sin(\delta_0 - \gamma)$$

Therefore,

$$\frac{2H}{\omega_{\text{Re}}} \frac{d^2 \Delta\delta}{dt^2} + \frac{D}{\omega_{\text{Re}}} \frac{d\Delta\delta}{dt} = -P_M (\cos(\delta_0 - \gamma))\Delta\delta \tag{7}$$

Or

$$\frac{2H}{\omega_{\text{Re}}} \frac{d^2 \Delta\delta}{dt^2} + \frac{D}{\omega_{\text{Re}}} \frac{d\Delta\delta}{dt} + P_M (\cos(\delta_0 - \gamma))\Delta\delta = 0 \tag{8}$$

Now define

$$P_S = P_M \cos(\delta_0 - \gamma) \tag{9}$$

What is it?

To answer this question, observe:

$$P_e = P_M \sin(\delta - \gamma) \tag{10}$$

$$\frac{dP_e}{d\delta} = P_M \cos(\delta - \gamma) \tag{11}$$

$$\left.\frac{dP_e}{d\delta}\right|_{\delta_0} = P_M \cos(\delta_0 - \gamma) \tag{12}$$

Therefore,

$$P_S = \left.\frac{dP_e}{d\delta}\right|_{\delta_0} = P_M \cos(\delta_0 - \gamma) \tag{13}$$

P_S is called the *synchronizing power coefficient*.

In regards to early swing instability (which is a nonlinear phenomena), the larger P_S is, the more stable will be the generator for a given disturbance.

This is true because P_S indicates the slope of the power-angle curve, and the higher this slope, the more decelerating energy is available to the machine for a given fault. This idea is illustrated in Fig. 2.

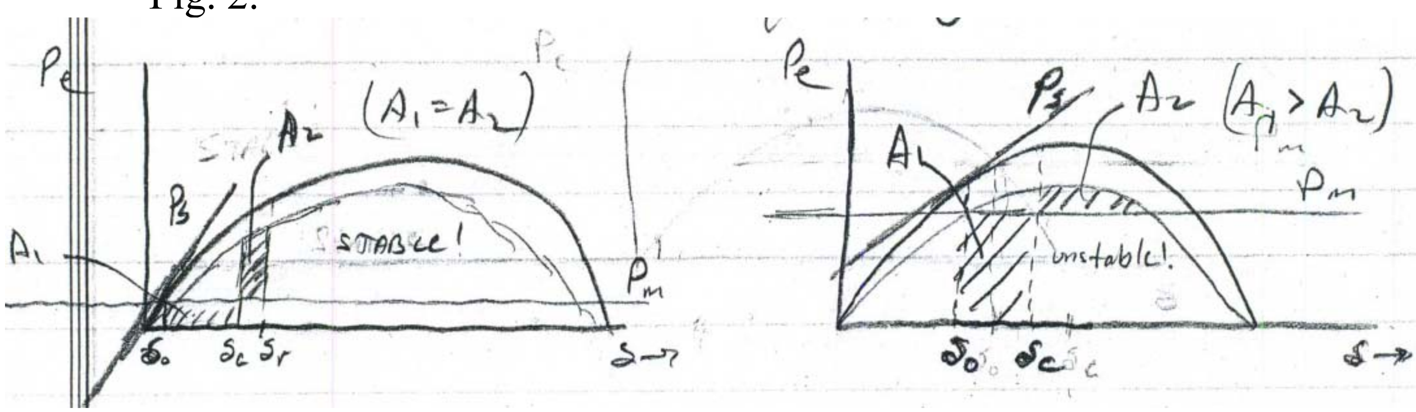

Fig. 2

But let's see what it means for "small signal instability," which is characterized by the eigenvalues (roots) of the system differential equation transformed to the s-domain through LaPlace transforms.

Substituting (13) into (8) results in

$$\frac{2H}{\omega_{\text{Re}}}\frac{d^2\Delta\delta}{dt^2}+\frac{D}{\omega_{\text{Re}}}\frac{d\Delta\delta}{dt}+P_S\Delta\delta=0 \qquad (14)$$

Taking the LaPlace transform (assuming all initial conditions are 0), we obtain

$$\frac{2H}{\omega_{\text{Re}}}s^2\Delta\delta(s)+\frac{D}{\omega_{\text{Re}}}s\Delta\delta(s)+P_S\Delta\delta(s)=0 \qquad (15)$$

Eliminating $\Delta\delta(s)$, we obtain the system's characteristic equation:

$$\frac{2H}{\omega_{\text{Re}}}s^2+\frac{D}{\omega_{\text{Re}}}s+P_S=0 \qquad (16)$$

(Eqt. 3.7)

Solving using the quadratic formula, we get

$$s=-\frac{D}{4H}\pm\frac{1}{2}\sqrt{\frac{D^2}{4H^2}-\frac{2P_S\omega_{\text{Re}}}{H}} \qquad (17)$$

Pulling $\omega_{\text{Re}}/2H$ out of the radical, we have

$$s=-\frac{D}{4H}\pm\frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2-\frac{8P_SH}{\omega_{\text{Re}}}} \qquad (18)$$

(Eqt. 3.8)

We can make some observations about (18), as follows:

1. If D=0, then

$$s=\pm\frac{\omega_{\text{Re}}}{4H}\sqrt{-\frac{8P_SH}{\omega_{\text{Re}}}}=\pm\sqrt{-\frac{\omega_{\text{Re}}P_S}{2H}},$$

   or

$$s=\pm j\sqrt{\frac{\omega_{\text{Re}}P_S}{2H}} \qquad (19)$$

a. Observe in (19) that if $P_S>0$, (a) any response to a small disturbance will be oscillatory, and (b) the oscillatory frequency becomes lower as H becomes larger.

b. Observe in (19) that if $P_S<0$, then

$$s = \pm\sqrt{\frac{\omega_{\text{Re}} \mid P_S \mid}{2H}} = \pm\sigma \tag{20}$$

and any response is unstable.

Figures 3, 4 illustrate, for both situations $P_S>0$, $P_S<0$, respectively, the pole (eigenvalue) locations in the s-plane and the operating point location on the power-angle curve.

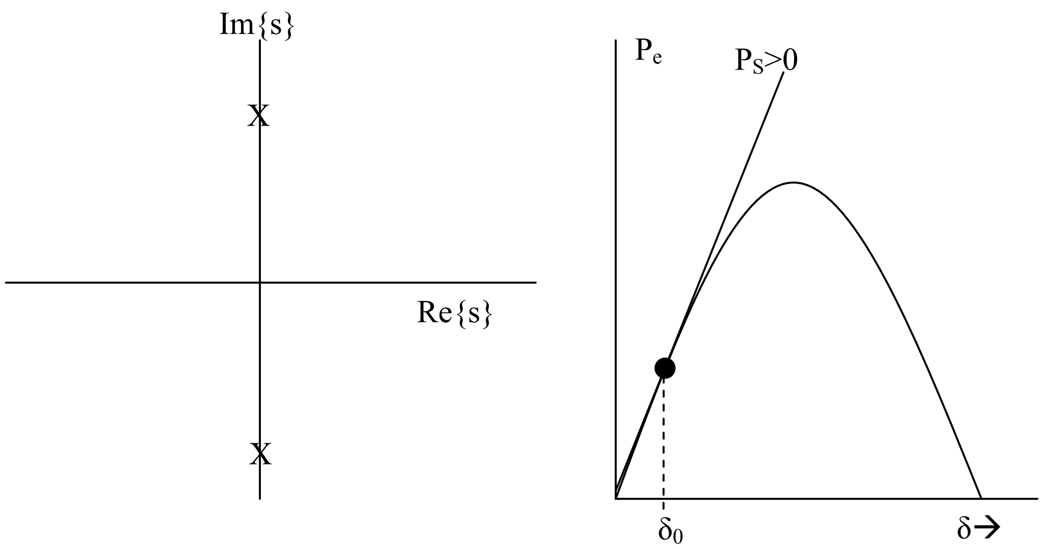

Fig. 3: $P_S>0$

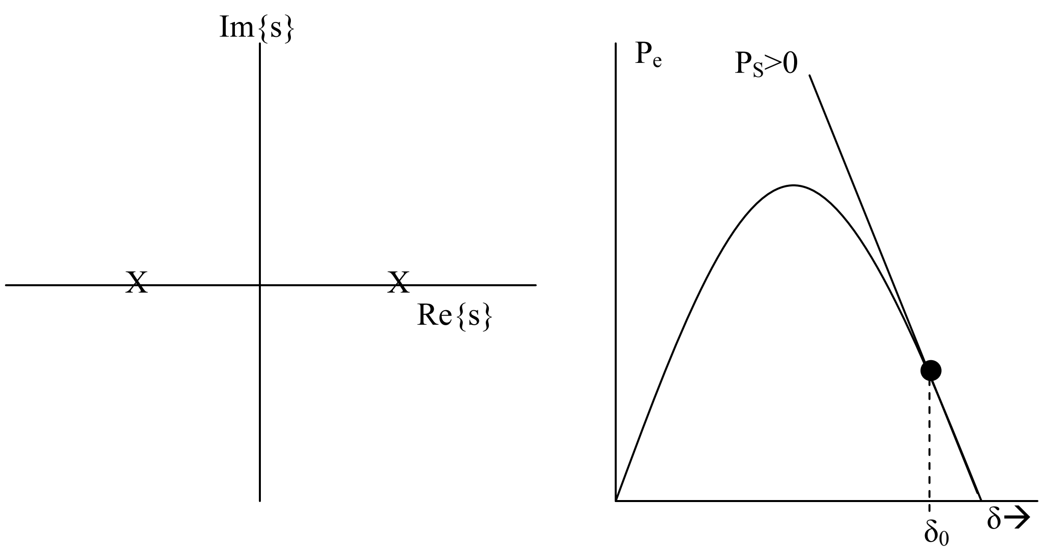

Fig. 4: P_S<0

In Fig. 3, the oscillatory system is characterized by purely imaginary poles (left) and a stable operating point (right). In Fig. 4, the unstable system is characterized by the RHP-pole (left-hand-side) and an unstable equilibrium point (right-hand-side).

2. If D≠0, then

$$s = -\frac{D}{4H} \pm \frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 - \frac{8P_S H}{\omega_{\text{Re}}}} \qquad (18)$$

Let's look at the most positive root (and so we will use "+" sign before the radical, and we ensure the contribution from the second term inside the radical is positive, i.e., P_S<0) and ask what are the conditions under which it can be in the right-half-plane, that is:

$$s = -\frac{D}{4H} + \frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}}} \overset{?}{>} 0$$

$$\Rightarrow \frac{\omega_{\text{Re}}}{4H}\sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}}} \overset{?}{>} \frac{D}{4H}$$

$$\Rightarrow \sqrt{\left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}}} \overset{?}{>} \frac{D}{\omega_{\text{Re}}}$$

$$\Rightarrow \left(\frac{D}{\omega_{\text{Re}}}\right)^2 + \frac{8|P_S|H}{\omega_{\text{Re}}} \overset{?}{>} \left(\frac{D}{\omega_{\text{Re}}}\right)^2$$

$$\Rightarrow \frac{8|P_S|H}{\omega_{\text{Re}}} \overset{?}{>} 0$$

The above relation must be true, given that $P_S<0$. Because the above relation is independent of damping, we conclude that if $P_S<0$, the system must be unstable.

2.0 Multi-machine case (Section 3.4)
(We will come back to sections 3.2 and 3.3.1)

Recall that for a generator connected to an infinite bus, we found that the swing equation is

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d^2\delta}{dt^2}+\frac{D}{\omega_{\mathrm{Re}}}\frac{d\delta}{dt}=P_m-P_e \tag{21}$$

where

- $P_e = P_M \sin(\delta-\gamma)$
- $P_M = |E||V||Y_{12}|$
- $Y_{12} = |Y_{12}|\angle\theta_{12}$

Letting $\delta=\delta_0+\Delta\delta$ and linearizing, we find that

$$\frac{2H}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta}{dt^2}+\frac{D}{\omega_{\mathrm{Re}}}\frac{d\Delta\delta}{dt}-P_S\Delta\delta=0 \tag{22}$$

where

$$P_S=\left.\frac{dP_e}{d\delta}\right|_{\delta_0}=P_M\cos(\delta_0-\gamma) \tag{13}$$

Let's now consider the multi-machine system assuming:

- Classical models
- Network reduced to only internal generator nodes

For generator i, we have that the swing equation is

$$\frac{2H_i}{\omega_{\mathrm{Re}}}\frac{d^2\Delta\delta_i}{dt^2}+\frac{D_i}{\omega_{\mathrm{Re}}}\frac{d\Delta\delta_i}{dt}=P_{m_i}-P_{e_i} \tag{23}$$

where

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_i + \delta_j)$$

$$= E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij}) \qquad (24)$$

where $\delta_{ij}=\delta_i-\delta_j$.

In (24), all voltages E_i, E_j, and all Y-bus elements Y_{ij} are magnitudes.

Now let's consider a small change in the angle of machine i: $\delta_i=\delta_{i0}+\Delta\delta_j$.

The left-hand-side of (24) is precisely as in the case of the single generator vs. infinite by s case. But what happened to the right-hand-side? Now the right-hand-side is, by (23), $P_{m_i} - P_{e_i}$.

- P_{mi} is unaffected by $+\Delta\delta_j$, but
- P_{ei} is affected by it.

Recall $\delta_{ij}=\delta_i-\delta_j$. We consider a small change in rotor angle at generator i. To be more general, we also allow a small change in generator j (but generator j will not change as a result of the generator i change; they are are independent changes and we could just as well have only one of them.

$\delta_i=\delta_{i0}+\Delta\delta_j$ $\qquad\qquad$ $\delta_j=\delta_{j0}+\Delta\delta_j$

Recalling that

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij}) \qquad (25)$$

we need to see what happens to the cos term for the small change in angle.

We know from trigonometry that

$$\cos(x-y)=\sin x\sin y+\cos x\cos y$$

Then

$$\cos(\theta_{ij}-\delta_{ij})=\sin\theta_{ij}\sin\delta_{ij}+\cos\theta_{ij}\cos\delta_{ij} \quad (26)$$

Application of (26) to (25) yields:

$$P_{ei}=E_i^2G_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}E_iE_j\left\{Y_{ij}\sin\theta_{ij}\sin\delta_{ij}+Y_{ij}\cos\theta_{ij}\cos\delta_{ij}\right\}$$

$$=E_i^2G_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}E_iE_j\left\{B_{ij}\sin\delta_{ij}+G_{ij}\cos\delta_{ij}\right\}$$

(27)

(Eqt. 3.21)

Now we need to linearize the $\cos\delta_{ij}$ and $\sin\delta_{ij}$ terms using $\delta_{ij}=\delta_{ij0}+\Delta\delta_{ij}$.

From Taylor series with first order term only,

$$\sin\delta_{ij}=\sin(\delta_{ij0}+\Delta\delta_{ij})=\sin\delta_{ij0}+\Delta\delta_{ij}\cos\delta_{ij0} \quad (28)$$

$$\cos\delta_{ij}=\cos(\delta_{ij0}+\Delta\delta_{ij})=\cos\delta_{ij0}-\Delta\delta_{ij}\sin\delta_{ij0} \quad (29)$$

Substituting (28) and (29) into (27), we get

$$P_{ei}=E_i^2G_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}E_iE_j\left\{B_{ij}\left(\sin\delta_{ij0}+\Delta\delta_{ij}\cos\delta_{ij0}\right)\right.$$

$$\left.+G_{ij}\left(\cos\delta_{ij0}-\Delta\delta_{ij}\sin\delta_{ij0}\right)\right\}$$

(30)

Now collect terms in $\Delta\delta_{ij}$:

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j \left\{ B_{ij} \sin\delta_{ij0} + G_{ij} \cos\delta_{ij0} \right\}$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j \left\{ B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0} \right\} \Delta\delta_{ij}$$

$$= P_{mi} + \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j \left\{ B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0} \right\} \Delta\delta_{ij}$$

(31a)

Recall that the right-hand-side of the swing equation is P_{mi}-P_{ei}. Equation (31a) can be rewritten then as (31b)

$$P_{ei} - P_{mi} = \sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j \left\{ B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0} \right\} \Delta\delta_{ij} \tag{31b}$$

therefore the swing equation (23), which is

$$\frac{2H_i}{\omega_{\mathrm{Re}}} \frac{d^2 \Delta\delta_i}{dt^2} + \frac{D_i}{\omega_{\mathrm{Re}}} \frac{d\Delta\delta_i}{dt} = P_{m_i} - P_{e_i} \tag{23}$$

becomes

$$\frac{2H_i}{\omega_{\mathrm{Re}}} \frac{d^2 \Delta\delta_i}{dt^2} + \frac{D_i}{\omega_{\mathrm{Re}}} \frac{d\Delta\delta_i}{dt} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} E_i E_j \left\{ B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0} \right\} \Delta\delta_{ij}$$

(32)

Define everything inside the expression within the summation of (32), except $\Delta\delta_{ij}$, as P_{Sij}, that is

$$P_{Sij} = E_i E_j \{B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0}\} \tag{33}$$

Then (32) becomes

$$\frac{2H_i}{\omega_{\text{Re}}} \frac{d^2\Delta\delta_i}{dt^2} + \frac{D_i}{\omega_{\text{Re}}} \frac{d\Delta\delta_i}{dt} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} P_{Sij}\Delta\delta_{ij} \tag{34}$$

Given the mechanical power is constant, the right-hand-side of (34) gives the negative of the change in electric power out of the machine due to the small changes $\Delta\delta_{ij}$, that is

$$\Delta P_{ei} = \sum_{\substack{j=1 \\ j\neq i}}^{n} P_{Sij}\Delta\delta_{ij} \tag{35}$$

(Eqt. 3.23)

What is P_{Sij}? We answer this question by observing that the power flowing from generator internal node i to generator internal node j is

$$P_{ij} = E_i E_j \{B_{ij} \sin\delta_{ij} + G_{ij} \cos\delta_{ij}\} \tag{36}$$

Differentiating, we get

$$\frac{\partial P_{ij}}{\partial \delta_{ij}} = E_i E_j \{B_{ij} \cos\delta_{ij} - G_{ij} \sin\delta_{ij}\} \tag{37}$$

Evaluating at δ_{ij0}, we get

$$\left.\frac{\partial P_{ij}}{\partial \delta_{ij}}\right|_{\delta_{ij0}} = E_i E_j \{B_{ij} \cos\delta_{ij0} - G_{ij} \sin\delta_{ij0}\} = P_{Sij} \tag{38}$$

(Eqt. 3.24)

Note that if bus j is the infinite bus, and neglecting resistance, we have that

$$P_{Sij} = E_i E_j B_{ij} \cos\delta_{ij0}$$

which is the same as the synchronizing power coefficient in the infinite bus case (we called it P_S).