INTRODUCTION TO THE 1989 IEEE/PES SYMPOSIUM ON EIGENANALYSIS AND FREQUENCY DOMAIN METHODS FOR SYSTEM DYNAMIC PERFORMANCE

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Abstract - To make the Symposium papers more accessible to non-specialists, a tutorial review is made of the basic terminology and mathematics that underlie them. Featured topics includes s-domain treatment of elementary dynamic systems, fundamental aspects of eigenanalysis, participation factors, and singular values.

Keywords: dynamic system, transfer function, eigenvalue, participation factor, singular value.

I. INTRODUCTION

This symposium presents contemporary mathematical methods and tools that are becoming essential to the planning, analysis, and control of modern power systems. The increasing complexity of these systems has mandated much closer inspection of such issues as intrinsic stability, dynamic interactions, and model validity. Use of even well-established tools for this requires at least a basic familiarity with the underlying mathematics, however. The necessary concepts are fairly straightforward, and are fundamental material from differential equations and/or automatic control. Even so, there is a good deal of this material, and it is well-laced with special terms and notation. It is hoped that this Introduction will help the non-specialist to overcome this.

II. SOME FUNDAMENTAL NOMENCLATURE

The systems considered here are commonly represented in the form

$$\dot{x} = Ax + Bu \ , \tag{1A}$$

Basic model

$$y = Cx + Du \tag{1B}$$

where $\dot{x}$ denotes differentiation with respect to time. Variables u and y are respectively the input and the output of the system; x, the internal state of the system, is usually taken to be a vector of n elements (n being the order of the system differential equation). System matrices A,B,C,D are fixed under present assumptions, so the system itself is termed "LTI" (linear time-invariant). Were one or more of these matrices time-varying, the system would be "LTV".

Systems are also classified according to the number of (scalar) inputs and outputs. Thus, if both u and y contain just one element, the system is "SISO" (single-input, single-output). If u and/or y contain multiple elements then the system is MIMO, MISO, or SIMO.

III. DYNAMICS OF LOW-ORDER SYSTEMS

The papers presented in this symposium build, with various strategies and objectives, upon the same central aspect of dynamic systems analysis. This is the ability to predict the future time-domain behavior of an LTI system through relatively simple calculations in the "complex frequency" domain [1-3]. An elementary example of this is provided by a system obeying the first-order differential equation

$$\dot{x} = \sigma x + u(t) \tag{2A}$$

and with the output

$$y = Cx \ . \tag{2B}$$

All quantities are scalar in this very reduced case. Suppose that the system starts at rest, with initial condition $x(t_0)=x_0=0$ at time $t_0=0$. Then equation (2A) Laplace transforms as

$$sX(s) = \sigma X(s) + U(s) \tag{3A}$$

and

$$Y(s) = \frac{CU(s)}{s-\sigma} \triangleq G(s)U(s) \tag{3B}$$

where

$$G(s) = \frac{C}{s-\sigma} \tag{3C}$$

is the transfer function for the system. (The qualified equality "$\triangleq$" is used to indicate the introduction of newly defined terms.) Response to particular inputs u(t) can now be determined by calculating the associated U(s) and inverting G(s)U(s) to time domain. For example, a unit impulse produces U(s)=1 and

$$y(t) = C\exp(\sigma t) \ . \tag{4}$$

For u(t) a step of height $\bar{U}$, $U(s)=\bar{U}/s$ and

$$Y(S) = \frac{C\bar{U}}{s(s-\sigma)} \tag{5A}$$

$$= \frac{C\bar{U}/\sigma}{s} + \frac{-C\bar{U}/\sigma}{s-\sigma} \triangleq \frac{K_0}{s} + \frac{K_1}{s-\sigma} \ , \tag{5B}$$

$$y(t) = K_0 + K_1\exp(\sigma t) \tag{5C}$$

$$= (C\bar{U}/\sigma)[1 - \exp(\sigma t)] \ . \tag{5D}$$

Equation (5B) provides a partial-fraction expansion for Y(s). Each K_i weights a term of form $1/(s-\lambda_i)$, and is said to be the residue associated with λ_i. The λ_i are generally referred to (not quite interchangeably [3]) as either the eigenvalues of $y(t)$ or the poles of $Y(s)$. It should be noted that some of the λ_i are associated with $u(t)$, not the dynamic system itself.

System eigenvalues are good indicators of system characteristics, regardless of the input. Clearly, if σ in (2) is positive, the system is intrinsically unstable and a very unusual input would be needed to prevent $x(t)$ from growing without bound. (Automatic control systems are designed to produce such inputs.) As a somewhat more general case, consider a second-order system having as its transfer function

$$G(s) = \frac{K_R+jK_I}{s-\sigma-j\omega} + \frac{K_R-jK_I}{s-\sigma+j\omega} \tag{6A}$$

$$= \frac{2K_Rs - 2(K_R\sigma+K_I\omega)}{s^2 - 2\sigma s + (\sigma^2+\omega^2)} \tag{6B}$$

$$\triangleq \frac{N(s)}{D(s)} \tag{6C}$$

where $\lambda_{1,2} = \sigma \pm j\omega$ are the system eigenvalues, and $K_{1,2} = K_R+jK_I$ are the associated residues. Complex λ_i or K_i necessarily occur in conjugate pairs, since the coefficients of both $N(s)$ and $D(s)$ are real numbers. The eigenvalues are roots for the denominator polynomial $D(s)$ (which is also termed the characteristic polynomial for the system). That is, they are those special values of s that satisfy the characteristic equation

$$D(s) = 0 \ . \tag{7}$$

Suspending some fine distinctions, they are also the poles of $G(s)$. The roots of the numerator polynomial, $N(s)$, are termed the zeros of $G(s)$.

The characteristic polynomial for a second-order system is often written as

$$D(s) = s^2 - 2\sigma s + (\sigma^2+\omega^2) \tag{8A}$$

$$= s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{8B}$$

The system response for an impulse input is then of form

$$x(t) = K_1 \exp(\omega_1 t) + K_2 \exp(\omega_2 t) \tag{9A}$$

$$= 2 \ |K| \ \exp(\sigma t) \ \cos(\omega-\theta) \tag{9B}$$

$$= 2 \ |K| \ \exp(-\zeta\omega_n t)\cos(\omega_d-\theta) \tag{9C}$$

$$\theta = \tan^{-1}(K_I/K_R) \ . \tag{9D}$$

Accessory quantities introduced in (10B) are

$$\omega_d = \omega \qquad \text{(damped natural frequency)} \tag{10A}$$

$$\omega_n = \sqrt{\sigma^2+\omega^2} \qquad \text{(undamped natural frequency)} \tag{10B}$$

$$\zeta = -\sigma/\omega_n \qquad \text{(damping ratio)} \tag{10C}$$

Normalized curves for system response in terms of these parameters are provided in basic texts dealing with control systems or system dynamics [1,2].

IV. BASICS OF MODAL ANALYSIS

Extrapolating from the previous section, we conclude that an LTI system that is brought to an initial state $x(t_0)=x_0$ at time $t_0=0$ and then is allowed to "ring down" without further inputs will obey

$$x(s) = [sI-A]^{-1}x_0 \tag{11A}$$

in which I is the (n x n) identity matrix (having 1's on the main diagonal, but 0's everywhere else). The solution in time domain will be of form

$$x(t) = \sum_{i=1}^{n} R_i x_0 \exp(\lambda_i t) = \sum_{i=1}^{n} R_i x_o e^{\lambda_i t} \tag{11B}$$

where R_i is a residue matrix [3]. It is usual to associate each eigenvalue λ_i with a "mode" of the system (or of the system matrix A). For this reason the λ_i are sometimes called (complex) modal frequencies.

For some purposes it is sufficient to determine the λ_i and thus establish the intrinsic characteristics of the dynamic modes alone. Often, however, one wishes to construct a system model as a simply connected structure of 1st-order and 2nd-order dynamic blocks, such as those analyzed in the previous section. Both objectives can be attained as follows.

First seek a transformation $x = Tx_m$ so that the system differential equation (1A) becomes

$$\dot{x}_m = \Lambda x_m + B_m u, \tag{12A}$$

with

$$\Lambda = \text{diag}\ (\lambda_1) = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix} \tag{12B}$$

(If this is unsuccessful then Λ becomes more complicated [3].) Now

$$\dot{x} = T\dot{x}_m = ATx_m + Bu \tag{13C}$$

so, in the modal coordinates defined by matrix T,

$$\dot{x}_m = (T^{-1}AT)x_m + (T^{-1}B)u \tag{13D}$$

$$= \Lambda x_m + B_m u. \tag{13E}$$

In order to find the eigenvalues themselves, first let T_j be column j of T; then

$$AT = T\Lambda \rightarrow AT_j = \lambda_j T_j \rightarrow (A-\lambda_j I)T_j = 0. \tag{14}$$

Since T_j is not identically the zero vector but is orthogonal to all row vectors of $(A-\lambda_j I)$, it must be that $(A-\lambda_j I)$ is of less than full rank n. Thus

$$\det\ (A-\lambda_j I) = |A-\lambda_j I| = D(\lambda_j) = 0 \tag{15}$$

where $D(\cdot)$ here is the characteristic polynomial. It is equations (14) and (15) that must be solved, directly or otherwise, for the eigenvalues λ and the (right) eigenvectors T_j.

At this point one can readily devise an additional transformation that converts Λ into a block-diagonal A-matrix, with each block being either 1st or 2nd order. Corresponding changes will, of course, be needed in B,C,D. The resulting block-modal "realization" of the system equations is particularly useful for theoretical development, numerical simulation, control system design, and model identification.

The reader should be aware that the terminology concerning system modes is not consistent throughout the literature. Some authors associate a mode with each λ_j, for a total of n modes. Other authors count each block in the realization described above as a single mode, but qualify it as either real or complex, or according to its order. Power system engineers, having to deal most with complex "swing modes", tend to favor the latter terminology.

V. EIGENVECTORS AND PARTICIPATION FACTORS

There is important information to be extracted from the transformation matrix T and from its inverse, T^{-1}. Some accessory notation here is customary, and useful. Let

$$T = [p_1 \; . \; . \; . \; p_n] = P, \; T^{-1} = \begin{bmatrix} q_1^T \\ \cdot \\ \cdot \\ \cdot \\ q_n^T \end{bmatrix} = Q^T. \tag{16}$$

Then

$$AP_i = \lambda_i P_i, \; q_i^T A = \lambda_i q_i^T \tag{17}$$

where p_i, q_i^T are, respectively, right and left eigenvectors of A. The details of equation (11B), describing system ringdown from x_0, can be now found as follows:

$$\begin{aligned} x(s) &= [sI-A]^{-1}x_0 = PP^{-1}[sI-A]^{-1}PP^{-1}x_0 \\ &= P[sI-\Lambda]^{-1}P^{-1}x_0 \qquad\qquad (18) \\ &= \sum_{i=1}^{n} \frac{p_i(q_i^T x_0)}{s-\lambda_i} = \sum_{i=1}^{n} \frac{(q_i^T x_0)p_i}{s-\lambda_i}\;, \end{aligned}$$

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} [(q_i^T x_0)\exp(\lambda_i t)]p_i \qquad\qquad (19) \\ &= \sum_{i=1}^{n} R_i x_0 \exp(\lambda_i t) \end{aligned}$$

(note that $q_i^T x_0$ is a scalar quantity).

Equation (19) fosters the following observations:

a) Vector q_i^T determines the influence of x_0 upon mode i.

b) Vector p_i determines the distribution of mode i among the components of x(t) (the "mode shape").

Eigenvectors are particularly valuable for determining the phase aspects of mode shape. Scaling uncertainties greatly reduce their usefulness for assessing magnitude relations, however. The underlying transformation between x and x_m requires $Q^T = P^{-1}$, by which

$$\begin{aligned} q_j^T p_i &= 1 \qquad \text{if } i = j, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{20}$$

This still leaves p_i indeterminant with regard to length and directional sense. Arbitrary scaling rules can resolve this, but they leave a more important problem: effects due to relative scaling of the state variables themselves. An alternative is to inspect

$$P_{ki} = p_{ki} q_{ki} \tag{21}$$

where p_{ki} and q_{ki} are the kth elements of p_i and q_i. The "participation factors" P_{ki} are dimensionless measures of state x_k in mode i. From (19), P_{ki} is also element kk of R_i, and thus represents the residue that is produced for mode i by a unit impulse in element i of x_0.

VI. SINGULAR VALUES

The theory and numerical algorithms for dynamic systems analysis make extensive use of "singular value analysis." Parallel to eigenanalysis, this is based upon a matrix diagonalization in the general form

$$F = U\Sigma V. \tag{22}$$

The singular-value decomposition of F includes the following features:

a) Σ is a diagonal matrix whose entries are the singular values σ_i of F, usually ordered by descending magnitude.

b) Matrix F need not be square.

c) Matrices U, V are orthogonal, and are constructed from the eigenvectors of FF^T, F^TF respectively.

Singular value σ_i for a matrix F is the nonnegative square root of the eigenvalue $\lambda_i[F^TF]$ for F^TF. For a square matrix A, we have a special interest in the particular singular values

$$\begin{aligned} \bar{\sigma}[A] &= \sqrt{\lambda_{max}[A^TA]}\;, \\ \underline{\sigma}[A] &= \sqrt{\lambda_{min}[A^TA]}\;. \end{aligned} \tag{23}$$

These have the following useful properties:

$$\underline{\sigma}[A] \le |\lambda_i[A]| \le \bar{\sigma}\,[A] \tag{24}$$

$$\max_{\|x\|=1} \|Ax\| = \bar{\sigma}[A] \tag{25A}$$

$$\min_{\|x\|=1} \|Ax\| = \underline{\sigma}[A] \tag{25B}$$

where $\|x\|$ is the Euclidian norm $(x_1^2 + \ldots + x_n^2)^{1/2}$ of x. The inequalities in (24) can be used to establish magnitude bounds upon system eigenvalues, which can be related to uncertainties in system parameters through (25). This is a very important issue in the engineering of major control systems.

More generally, the singular values of a matrix provide a sharp indicator of its effective rank, and are a useful guide to the design or execution of high performance numerical algorithms. They are rapidly becoming a common tool in the mathematical treatment of dynamic systems.

VII. CONCLUDING REMARKS

This Introduction has provided a rather brief tour of vocabulary and mathematics that is fundamental to the treatment of contemporary power systems. For timely details on their application, ramifications, and extensions we shall turn to the Symposium authors.

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