

Flux-Linkage State-Space Model (Section 4.12)

One limitation of the current state-space model is that the effects of saturation are difficult to represent.

This is because saturation occurs in the mutual fluxes, since mutual flux exists in the core material, which is quite saturable (i.e., the permeability decreases with high currents), whereas leakage flux exists largely external to the core material, i.e., in the air, which has permeability that is nearly constant with current.

So we will develop the flux-linkage state-space model, which uses the λ 's as the state variables rather than the currents.

Our approach will follow four steps:

1. Develop d- and q- axis currents as function of λ 's and mutual fluxes λ_{AD} and λ_{AQ} (we will define these shortly).
2. Develop state equations for λ 's by substituting currents (step 1) into voltage equations.
3. Develop the torque equation in terms of flux linkages.
4. Approximate the effects of saturation.

Step 1: Develop d- and q- axis currents as function of λ 's and mutual fluxes λ_{AD} and λ_{AQ} :

(This comes partly from Section 4.11 and partly from 4.12)

We will take three sub-steps here.

- Step 1a: d-axis currents
- Step 1b: q-axis currents
- Step 1c: put them together

Step 1a: d-axis currents:

Recall that in per-unit, all d-axis mutual flux linkages are equal.

As we have seen, these d-axis mutual fluxes can be expressed as the total flux less the leakage flux, and we call it λ_{AD} :

$$\lambda_{AD} = \lambda_d - l_d i_d = \lambda_F - l_F i_F = \lambda_D - l_D i_D \quad (4.110)$$

This says the following:

- If in each circuit on the d-axis,
- the pu leakage flux linkage is subtracted from the pu main flux linkage,
- the remaining flux linkage is the same for all other circuits coupled to it.

Solving (4.110) for each of the currents, we obtain:

$$\begin{aligned} i_d &= \frac{1}{l_d} (\lambda_d - \lambda_{AD}) \\ i_F &= \frac{1}{l_F} (\lambda_F - \lambda_{AD}) \\ i_D &= \frac{1}{l_D} (\lambda_D - \lambda_{AD}) \end{aligned} \quad (4.118)$$

Now this appears to be good enough to allow us to go to step 2, and substitute the currents into the voltage equations; however, we have introduced an extra variable, λ_{AD} , which is actually dependent on the other flux linkages.

So let's see if we can express λ_{AD} in terms of the other fluxes:

$$\lambda_{AD} = \lambda_d - l_d i_d$$

Expand the total flux term, λ_d , as a function of the currents. From eq. 4.20' (pg 20 of "macheqts.doc"), we have that (in pu):

$$\lambda_d = L_d i_d + kM_F i_F + kM_D i_D$$

Making this substitution results in

$$\lambda_{AD} = [L_d i_d + kM_F i_F + kM_D i_D] - l_d i_d$$

Now combining the two terms containing i_d results in:

$$\lambda_{AD} = [L_d - l_d] i_d + [kM_F] i_F + [kM_D] i_D \quad (*)$$

where we recognize each term in brackets as L_{AD} , which was defined in the "perunitization" notes (pg 29) as the per-unit value of any d-axis mutual inductance, i.e.,

$$L_{AD} \equiv L_{mdu} = kM_{Fu} = kM_{Du} = M_{Ru}$$

Substitution of L_{AD} into (*) yields:

$$\lambda_{AD} = [L_{AD}] i_d + [L_{AD}] i_F + [L_{AD}] i_D = L_{AD} [i_d + i_F + i_D] \quad (4.111)$$

Substitution of (4.118) into (4.111) eliminates the currents:

$$\lambda_{AD} = L_{AD} \left[\frac{1}{l_d} (\lambda_d - \lambda_{AD}) + \frac{1}{l_F} (\lambda_F - \lambda_{AD}) + \frac{1}{l_D} (\lambda_D - \lambda_{AD}) \right]$$

Now divide through by L_{AD} and then collect terms in λ_{AD} :

$$\lambda_{AD} \left[\frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D} \right] = \frac{\lambda_d}{l_d} + \frac{\lambda_F}{l_F} + \frac{\lambda_D}{l_D} \quad (**)$$

Recognizing that the term in the brackets is the inverse of a parallel combination of the inductances, we define this term as:

$$\frac{1}{L_{MD}} = \left[\frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D} \right]$$

Making this substitution into (**) results in:

$$\frac{\lambda_{AD}}{L_{MD}} = \frac{\lambda_d}{l_d} + \frac{\lambda_F}{l_F} + \frac{\lambda_D}{l_D} \Rightarrow \lambda_{AD} = \frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MD}}{l_F} \lambda_F + \frac{L_{MD}}{l_D} \lambda_D \quad (4.120)$$

Step 1b: q-axis currents:

We repeat the above procedure for the q-axis currents. We will actually do this, despite its similarity to step 1a, to be sure we correctly deal with the G-circuit.

Recall that in per-unit, all q-axis mutual flux linkages are equal.

As we have seen, these d-axis mutual fluxes can be expressed as the total flux less the leakage flux, and we call it λ_{AQ} :

$$\lambda_{AQ} = \lambda_q - l_q i_q = \lambda_Q - l_Q i_Q = \lambda_G - l_G i_G \quad (4.110-Q)$$

This says the following:

- If in each circuit on the q-axis,
- the pu leakage flux linkage is subtracted from the pu main flux linkage,
- the remaining flux linkage is the same for all other circuits coupled to it.

Solving (4.110-Q) for each of the currents, we obtain:

$$\begin{aligned} i_q &= \frac{1}{l_q} (\lambda_q - \lambda_{AQ}) \\ i_Q &= \frac{1}{l_Q} (\lambda_Q - \lambda_{AQ}) \\ i_G &= \frac{1}{l_G} (\lambda_G - \lambda_{AQ}) \end{aligned} \quad (4.118-Q)$$

(similar to 4.123)

Now this appears to be good enough to allow us to go to step 2, and substitute the currents into the voltage equations; however, we have introduced an extra variable, λ_{AQ} , which is actually dependent on the other flux linkages.

So let's see if we can express λ_{AQ} in terms of the other fluxes.

$$\lambda_{AQ} = \lambda_q - l_q i_q$$

Expand the total flux term, λ_q , as a function of the currents. From eq. 4.20' (pg 20 of "macheqts.doc"), we have that (in pu):

$$\lambda_q = L_q i_q + kM_Q i_Q + kM_G i_G$$

Making this substitution results in

$$\lambda_{AQ} = [L_q i_q + kM_Q i_Q + kM_G i_G] - l_q i_q$$

Now combining the two terms containing i_q results in:

$$\lambda_{AQ} = [L_q - l_q] i_q + [kM_Q] i_Q + [kM_G] i_G$$

where we recognize each term in brackets as L_{AQ} , which was defined in the "perunitization" notes (pg 30) as the per-unit value of any q-axis mutual inductance, i.e.,

$$L_{AQ} \equiv L_{mqu} = kM_{Qu} = kM_{Gu} = M_{Yu}$$

so that:

$$\lambda_{AQ} = [L_{AQ}] i_q + [L_{AQ}] i_Q + [L_{AQ}] i_G = L_{AQ} [i_q + i_Q + i_G] \quad (4.111-Q)$$

(Similar to 4.112)

Substitution of (4.118-Q) into (4.111-Q) eliminates the currents:

$$\lambda_{AQ} = L_{AQ} \left[\frac{1}{l_q} (\lambda_q - \lambda_{AQ}) + \frac{1}{l_Q} (\lambda_Q - \lambda_{AQ}) + \frac{1}{l_G} (\lambda_G - \lambda_{AQ}) \right]$$

Now divide through by L_{AQ} and then collect terms in λ_{AQ} , resulting in:

$$\lambda_{AQ} \left[\frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G} \right] = \frac{\lambda_q}{l_q} + \frac{\lambda_Q}{l_Q} + \frac{\lambda_G}{l_G}$$

Recognizing that the term in the brackets is the inverse of a parallel combination of the inductances, we define it as:

$$\frac{1}{L_{MQ}} = \left[\frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G} \right]$$

Making this substitution results in:

$$\frac{\lambda_{AQ}}{L_{MQ}} = \frac{\lambda_q}{l_q} + \frac{\lambda_Q}{l_Q} + \frac{\lambda_G}{l_G} \rightarrow \lambda_{AQ} = \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G \quad (4.120-Q)$$

(similar to 4.121)

We refer to eqs. (4.120) and (4.120-Q) as the *auxiliary equations*.

$$\lambda_{AD} = \frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MD}}{l_F} \lambda_F + \frac{L_{MD}}{l_D} \lambda_D \quad (4.120)$$

Step 1c: Putting them together:

Now combine equations 4.118 with 4.118-Q, and write them all into a single matrix expression of currents in terms of the fluxes:

$$\begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ i_G \end{bmatrix} = \begin{bmatrix} \frac{1}{l_d} & 0 & 0 & -\frac{1}{l_d} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{l_F} & 0 & -\frac{1}{l_d} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{l_D} & -\frac{1}{l_D} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{l_q} & 0 & 0 & -\frac{1}{l_q} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{l_Q} & 0 & -\frac{1}{l_Q} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{l_G} & -\frac{1}{l_G} \end{bmatrix} \begin{bmatrix} \lambda_d \\ \lambda_F \\ \lambda_D \\ \lambda_{AD} \\ \lambda_q \\ \lambda_Q \\ \lambda_G \\ \lambda_{AQ} \end{bmatrix} \quad (4.124')$$

where we know from 4.120 and 4.120-Q that λ_{AD} and λ_{AQ} are “combination” states developed from the other flux-linkage states.

Step 2: Develop state equations for λ 's by substituting currents (step 1) into voltage equations (see Section 4.12.1)

The procedure here is:

- a. Start from a preliminary form of the voltage equation that contains flux linkage derivatives instead of current derivatives. This is convenient because it immediately provides us with the derivatives on the states we desire. The form that we will use is given by eq. 4.36 (where I have included the G-circuit voltage equation) in the text, which is

$$\begin{bmatrix} v_d \\ v_q \\ -v_F \\ v_D = 0 \\ v_Q = 0 \\ v_G = 0 \end{bmatrix} = - \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & 0 & r_D & 0 & 0 \\ 0 & 0 & 0 & 0 & r_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & r_G \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_F \\ i_D \\ i_Q \\ i_G \end{bmatrix} + \begin{bmatrix} -\omega\lambda_q \\ \omega\lambda_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{\lambda}_q \\ \dot{\lambda}_d \\ \dot{\lambda}_F \\ \dot{\lambda}_D \\ \dot{\lambda}_Q \\ \dot{\lambda}_G \end{bmatrix}$$

- b. Substitute for the currents using 4.124', and do some algebra.

Let's do the procedure for the d-axis flux linkage (as in the text) as well as the g-axis flux linkage.

d-axis flux linkage equation:

Step a:
$$v_d = -ri_d - \dot{\lambda}_d - \omega\lambda_q$$

Step b: From eq. 4.124', we have that:
$$i_d = \frac{\lambda_d}{l_d} - \frac{\lambda_{AD}}{l_d}$$

Substituting, we have:

$$v_d = -r\left(\frac{\lambda_d}{l_d} - \frac{\lambda_{AD}}{l_d}\right) - \dot{\lambda}_d - \omega\lambda_q = -r\frac{\lambda_d}{l_d} + r\frac{\lambda_{AD}}{l_d} - \dot{\lambda}_d - \omega\lambda_q$$

Solving for $d\lambda_d/dt$ results in: $\dot{\lambda}_d = -r\frac{\lambda_d}{l_d} + r\frac{\lambda_{AD}}{l_d} - \omega\lambda_q - v_d$

G-axis flux linkage equation:

Step a: $v_G = 0 = -r_G i_G - \dot{\lambda}_G$

Step b: From eq. 4.124', we have that: $i_G = \frac{\lambda_G}{l_G} - \frac{\lambda_{AQ}}{l_G}$

Substituting, we have:

$$0 = -r_G\left(\frac{\lambda_G}{l_G} - \frac{\lambda_{AQ}}{l_G}\right) - \dot{\lambda}_G = -r_G\frac{\lambda_G}{l_G} + r_G\frac{\lambda_{AQ}}{l_G} - \dot{\lambda}_G$$

Solving for $d\lambda_G/dt$ results in: $\dot{\lambda}_G = -r_G\frac{\lambda_G}{l_G} + r_G\frac{\lambda_{AQ}}{l_G}$

Summarize all equations

$$\dot{\lambda}_d = -\frac{r}{l_d}\lambda_d + \frac{r}{l_d}\lambda_{AD} - \omega\lambda_q - v_d \quad (4.126)$$

$$\dot{\lambda}_F = -\frac{r_F}{l_F}\lambda_F + \frac{r_F}{l_F}\lambda_{AD} + v_F \quad (4.128)$$

$$\dot{\lambda}_D = -\frac{r_D}{l_D}\lambda_D + \frac{r_D}{l_D}\lambda_{AD} \quad (4.129)$$

$$\dot{\lambda}_q = -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \lambda_{AQ} + \omega \lambda_d - v_q \quad (4.130)$$

$$\dot{\lambda}_Q = -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \lambda_{AQ} \quad (4.131)$$

$$\dot{\lambda}_G = -\frac{r_G}{l_G} \lambda_G + \frac{r_G}{l_G} \lambda_{AQ} \quad (4.131')$$

Step 3: Develop the torque equation in terms of flux linkages.

We must also write the torque equation as a function of the flux linkages.

Here, we return to the expression we developed for the torque corresponding to the power crossing the air gap. This was given as:

$$T_{e\phi u} = -\lambda_q i_d + \lambda_d i_q$$

When developing the current state-space equations, you recall that at this point we substituted for the flux linkages. Now, when developing the flux-linkage state-space equations, we substitute for the currents. Use 4.124' from above, which, for i_q and i_d , is:

$$i_d = \frac{\lambda_d - \lambda_{AD}}{l_d} \quad i_q = \frac{\lambda_q - \lambda_{AQ}}{l_q}$$

Substitution yields:

$$T_{e\phi u} = -\lambda_q \frac{\lambda_d - \lambda_{AD}}{l_d} + \lambda_d \frac{\lambda_q - \lambda_{AQ}}{l_q}$$

Performing the multiplication, and then gathering terms, results in

$$T_{e\phi u} = -\frac{1}{l_q} \lambda_d \lambda_{AQ} + \frac{1}{l_d} \lambda_q \lambda_{AD} + \left(\frac{1}{l_q} - \frac{1}{l_d} \right) \lambda_d \lambda_q \quad (4.132a)$$

A+F indicate that the leakage inductance is the same in the d- and q-axis, that is, $l_q=l_d$. I believe this is true for round-rotor machines but only approximate for salient-pole machines. Application of it results in:

$$T_{e\phi u} = -\frac{1}{l_q} \lambda_d \lambda_{AQ} + \frac{1}{l_d} \lambda_q \lambda_{AD} \quad (4.132b)$$

Recall the torque equation derived for the current-state-space model:

$$\dot{\omega} = \frac{T_m}{\tau_j} + \frac{1}{3\tau_j} [-T_{e\phi}] + \left[\frac{-D}{\tau_j} \right] \omega$$

Substitution of (4.132b) yields:

$$\dot{\omega} = \frac{T_m}{\tau_j} + \left[\frac{\lambda_{AQ}}{l_q 3\tau_j} \lambda_d - \frac{\lambda_{AD}}{l_d 3\tau_j} \lambda_q \right] + \left[\frac{-D}{\tau_j} \right] \omega \quad (4.133)$$

And so we may gather our equations for the flux-linkage model as:

$$\dot{\lambda}_d = -\frac{r}{l_d} \lambda_d + \frac{r}{l_d} \lambda_{AD} - \omega \lambda_q - v_d \quad (4.126)$$

$$\dot{\lambda}_F = -\frac{r_F}{l_F} \lambda_F + \frac{r_F}{l_F} \lambda_{AD} + v_F \quad (4.128)$$

$$\dot{\lambda}_D = -\frac{r_D}{l_D} \lambda_D + \frac{r_D}{l_D} \lambda_{AD} \quad (4.129)$$

$$\dot{\lambda}_q = -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \lambda_{AQ} + \omega \lambda_d - v_q \quad (4.130)$$

$$\dot{\lambda}_Q = -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \lambda_{AQ} \quad (4.131)$$

$$\dot{\lambda}_G = -\frac{r_G}{l_G} \lambda_G + \frac{r_G}{l_G} \lambda_{AQ} \quad (4.131')$$

$$\dot{\omega} = \frac{T_m}{\tau_j} + \left[\frac{\lambda_{AQ}}{l_q 3\tau_j} \lambda_d - \frac{\lambda_{AD}}{l_d 3\tau_j} \lambda_q \right] + \left[\frac{-D}{\tau_j} \right] \omega \quad (4.133)$$

$$\dot{\delta} = \omega - 1 \quad (4.102)$$

where:

$$\lambda_{AD} = \frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MD}}{l_F} \lambda_F + \frac{L_{MD}}{l_D} \lambda_D$$

$$\lambda_{AQ} = \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G$$

(called the “auxiliary equations) and

$$\frac{1}{L_{MD}} = \left[\frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D} \right]$$

$$\frac{1}{L_{MQ}} = \left[\frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G} \right]$$

The state variables are $\lambda_d, \lambda_F, \lambda_D, \lambda_q, \lambda_Q, \lambda_G, \omega$, and δ .

The forcing functions are v_d, v_q, v_F , and T_m (v_d and v_q depend on the currents, which in turn depend on the network loading – we will deal with this issue later!).

Recall that $\lambda_{AD}, \lambda_{AQ}$ represents the per-unit d-axis and q-axis mutual flux linkages, respectively, i.e.,

$$\lambda_{AD} = \lambda_d - l_d i_d = \lambda_F - l_F i_F = \lambda_D - l_D i_D \quad (4.110)$$

$$\lambda_{AQ} = \lambda_q - l_q i_q = \lambda_Q - l_Q i_Q = \lambda_G - l_G i_G \quad (4.110-Q)$$

and that it is only in them, the mutual flux, where saturation occurs. That is, of the total flux seen by each winding, saturation does not affect the leakage, only the mutual. Since these mutual fluxes occur in the same material, it is reasonable to assume the saturation characteristics are the same for all. This allows us to address saturation in a very simple way, through $\lambda_{AD}, \lambda_{AQ}$.

If we were to eliminate these two terms by making the appropriate substitution, then the effects of saturation would be distributed throughout our model, and its treatment would be very complex.

➔ So how do we deal with saturation?