Intro to Linear Programming

1.0 Introduction

The problem that we desire to address in this course is loosely stated below.

Given

- a number of generators make price-quantity offers to sell (each provides their individual supply function)
- a number of loads make price-quantity bids to buy (each provides their individual demand function)
- there is an electric network that imposes physical restrictions, including:
    - sum of generation=sum of demand
    - flows on circuits≤maximum flow for circuit

we want to maximize the social surplus

$$U(\underline{P})-C(\underline{P})$$

where $U(\underline{P})$ is the composite utility function for consumers, $C(\underline{P})$ is the composite cost function for suppliers, and $\underline{P}$ is the real power injection vector of the network nodes (positive for generation, negative for load).

It is sometimes convenient to recognize a couple of things about the form of optimization problems.

1. Maximization of a function is equivalent to minimization of the negated function, that is

max $f(\underline{x})$ is the same as min $(-f(\underline{x}))$

2. An inequality constraint may be equivalently written as

$$g(\underline{x}) \geq b \qquad \text{or} \qquad -g(\underline{x}) \leq b$$

And so we can say that in general, we want to solve a problem like this:

$$\begin{aligned} \min\ & f(\underline{x}) \\ s.t.\ & \underline{h}(\underline{x}) = \underline{c} \\ & \underline{g}(\underline{x}) \leq \underline{b} \end{aligned} \tag{1}$$

We have already seen how to do this when $f(\underline{x})$, $\underline{h}(\underline{x})$, and $g(\underline{x})$ are nonlinear but convex (form Lagrangian, take KKT conditions, solve for variables).

We now want to consider what happens when f, $\underline{h}$, and $\underline{g}$ are all linear functions in $\underline{x}$.

When this is the case, the optimization problem of (1), generally called a *mathematical program*, becomes a special type of mathematical program called a *linear program*.

We will motivate our interest with an example.

2.0 Example 1

$$\max \ f(\underline{x}) = 3x_1 + x_2$$
$$s.t. \quad x_1 + x_2 = 16$$
$$-x_1 + x_2 \le 4$$

Let's apply our standard procedure to this problem.

Form Lagrangian (ignoring inequality constraint):

$$F(\underline{x}, \lambda) = 3x_1 + x_2 - \lambda(\mathrm{x}_1 + x_2 - 16)$$

Apply first-order conditions:

$$\frac{\partial F}{\partial x_1} = 3 - \lambda = 0 \Rightarrow \lambda = 3$$

$$\frac{\partial F}{\partial x_2} = 1 - \lambda = 0 \Rightarrow \lambda = 1$$

$$\frac{\partial F}{\partial \lambda} = x_1 + x_2 - 16 = 0$$

The first two equations result in a contradiction, and the three equations taken together do not provide a solution for x_1 and x_2. Our procedure failed. What happened?

Let's see if we can graphically inspect the situation. Fig. 1 illustrates the contours of increasing objective function together with the equality constraint.

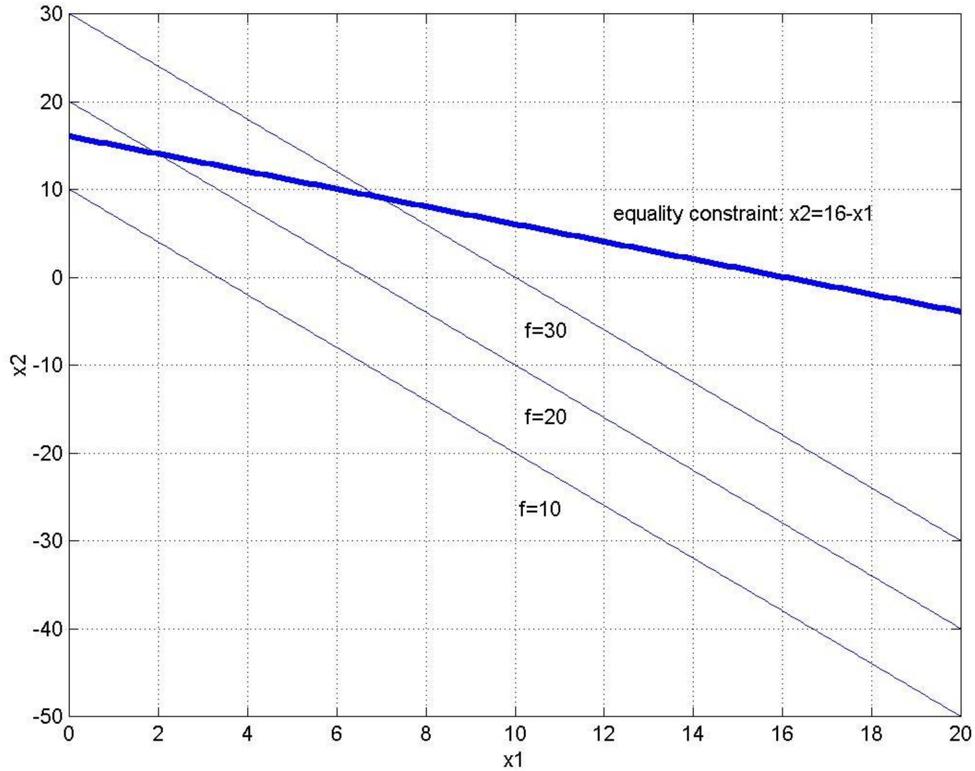

Fig. 1: Sample problem

In considering Fig. 1, we see that we can push *f* as far negative as we please (causing the contours to move down and to the left), minimizing it more and more, and there will always be an intersection point with the equality constraint. This shows that there is no solution to our problem. We might say that the solution is unbounded. This can happen with linear programs.

However, recall that we are considering only the equality constraint, i.e., the problem we considered for which there is no solution is:

$$\max \ f(\underline{x}) = 3x_1 + x_2$$
$$s.t. \quad x_1 + x_2 = 16$$

But let's consider our original, full problem, with the inequality constraint as well, repeated here for convenience.

$$\max \ f(\underline{x}) = 3x_1 + x_2$$
$$s.t. \quad x_1 + x_2 = 16$$
$$-x_1 + x_2 \leq 4$$

Form Lagrangian (this time with inequality constraint):

$$F(\underline{x}, \lambda) = 3x_1 + x_2 - \lambda(x_1 + x_2 - 16) + \mu(-x_1 + x_2 - 4)$$

Observe, in contrast to our previous work where we had an inequality constraint expressed as $\underline{g}(\underline{x}) \underline{\geq} \underline{b}$ and we subtracted the term corresponding to the inequality, now we have an inequality constraint expressed as $\underline{g}(\underline{x}) \underline{\leq b}$ and we add the term corresponding to it.

The KKT conditions then become:

$$\frac{\partial F}{\partial x_1} = 3 - \lambda - \mu = 0$$
$$\frac{\partial F}{\partial x_2} = 1 - \lambda + \mu = 0$$
$$\frac{\partial F}{\partial \lambda} = x_1 + x_2 - 16 = 0$$
$$\frac{\partial F}{\partial \mu} = -x_1 + x_2 - 4 = 0$$

$$\mu(-x_1 + x_2 - 4) = 0$$
$$\mu \geq 0$$

The left-hand set of equations is a set of linear equations we can solve as:

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 16 \\ 4 \end{bmatrix}$$

having solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 2 \\ 1 \end{bmatrix}$$

We plot it in Figure 2.

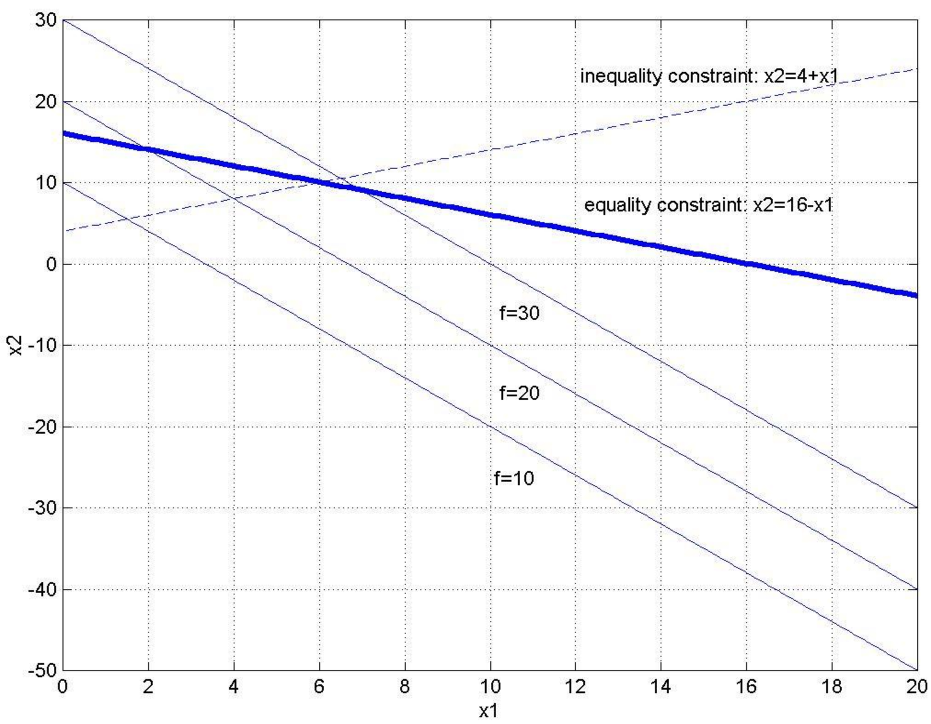

Fig. 2: With inequality constraint

The feasible region <u>associated with the inequality constraint</u> is below the dotted line.

The feasible region <u>for the problem</u> is below the dotted line and *on the thick one* (the equality constraint). So we see that the feasible region for the problem includes everything on the thick line to the right of the intersection point.

Now, recall that we are maximizing the objective. Therefore we want to choose a solution that has the largest possible value of f but is in the feasible region. This means, in terms of Fig. 3, we want to choose the contour that is highest (most upwards to the right) but feasible.

The location of the solution to the problem is immediate: $x_1=6$, $x_2=10$. The contour which passes through this point is not obvious from Fig. 3, but it is between $f=20$ and $f=30$, probably about $f=28$. We can get it exactly by evaluating f at the solution:

$$f(\underline{x}) = 3x_1 + x_2 = 3(6) + 10 = 28$$

which agrees exactly with our estimate.

We will call any constraint comprising the feasible region boundary an active constraint.

One important observation for this problem: *the solution occurred at a point where two active constraints intersect.*

Let's do another example. This time, we will work on a maximization problem.

3.0 Example 2 [1]

Resource allocation is a kind of problem where one desires to optimize some objective subject to constraints on a set of resources. It is a very common kind of problem in many different industries, including the electric power industry.

Consider a manufacturer of materials *X* and *Y*. Each materials produced requires a certain amount of time from three different people, and each person can allocate their time to producing item *X* or item *Y* or some combination of the two. A unit of material requires contributions from all three people, i.e., no individual can make either material on their own. The following table time contributions for each person to make each type of material.

Material	Person		
	1	2	3
X	40 min	24 min	20 min
Y	30 min	32 min	24 min

Each person works a standard 8 hour day, and so has a maximum amount of time per day of $8 \times 60 = 480$ minutes. The profit per unit of item *X* and *Y* is \$5 and \$8, respectively. We desire to maximize daily profits.

Let *x* and *y* denote the number of each item produced in a day. The problem will be, then, as follows:

$$\max\ f(x, y) = 5x + 8y$$

Subject to

$$40x + 30y \le 480 \quad \text{(person 1)}$$

$$24x + 32y \le 480 \quad \text{(person 2)}$$

$$20x + 24y \le 480 \quad \text{(person 3)}$$

$$x \ge 0$$

$$y \ge 0$$

The first three constraints are due to the limit on the *resource* (time) for each person. The 4^{th} and 5^{th} constraints are due to the fact that we cannot produce a negative number of items.

There are many possible solutions to this problem. Consider, for example, if we limit ourselves to producing only item *X* or item *Y*, but not both.

Produce only item *X*. In this case, *y=0* and the constraints become:

$$40x \le 480 \Rightarrow x \le 12 \text{ (person 1)}$$
$$24x \le 480 \Rightarrow x \le 20 \text{ (person 2)}$$
$$20x \le 480 \Rightarrow x \le 24 \text{ (person 3)}$$

To maximize the objective function, we want to produce the maximum amount of items we can. And so the solution is clearly x=12 (and y=0). The objective then, becomes 5(12)=$60.

Produce only item *Y*. In this case, *x=0* and the constraints become:

$$30y \le 480 \Rightarrow y \le 16 \text{ (person 1)}$$
$$32y \le 480 \Rightarrow y \le 15 \text{ (person 2)}$$
$$24y \le 480 \Rightarrow y \le 20 \text{ (person 3)}$$

Again, to maximize the objective function, we want to produce the maximum amount of items we can. And so the solution is clearly y=15 (and x=0). The objective then, becomes 8(15)=$120.

Conclusion: Producing only *Y* is better than producing only *X*, and if we have only these two options, we will produce only *Y*.

However, is it possible that we might make an even higher profit by producing some of each?

Fig. 3 plots the inequality constraints. We can identify the feasible region.

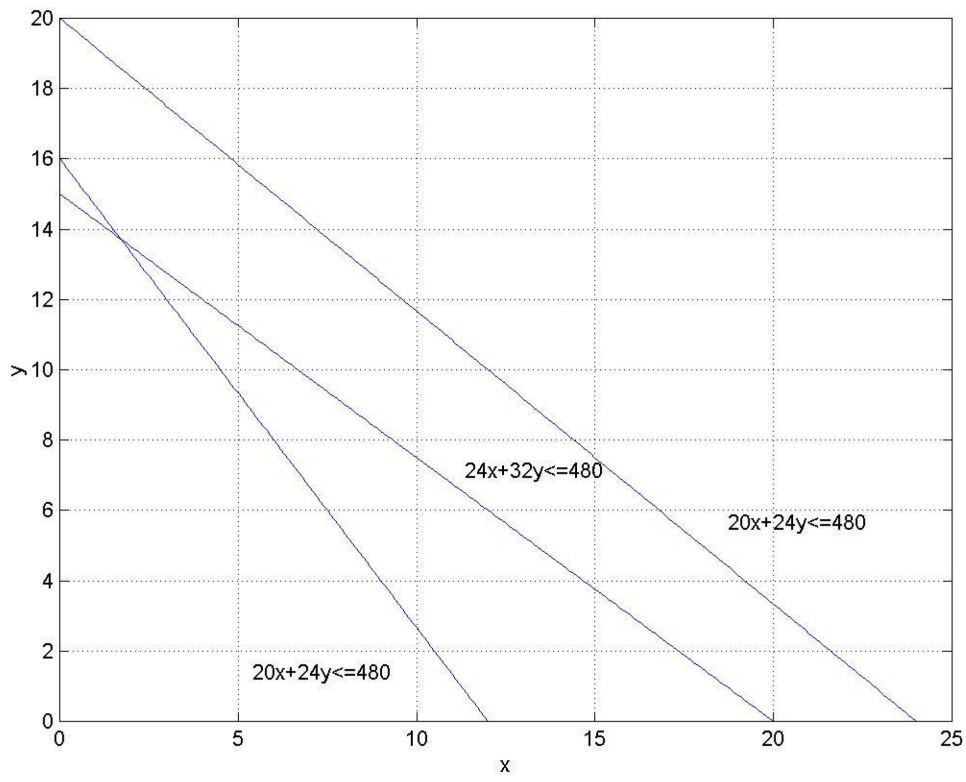

Fig. 3: Constraints for Example 2

It is now interesting to plot the contours of increasing objective function, that is, we will plot the function

$$f(x,y) = 5x + 8y$$

for increasing values of *f*.

$$f = 5x + 8y \Rightarrow y = \frac{f - 5x}{8}$$

We have plotted the contours for values of *f*=10, 20, …, 170.

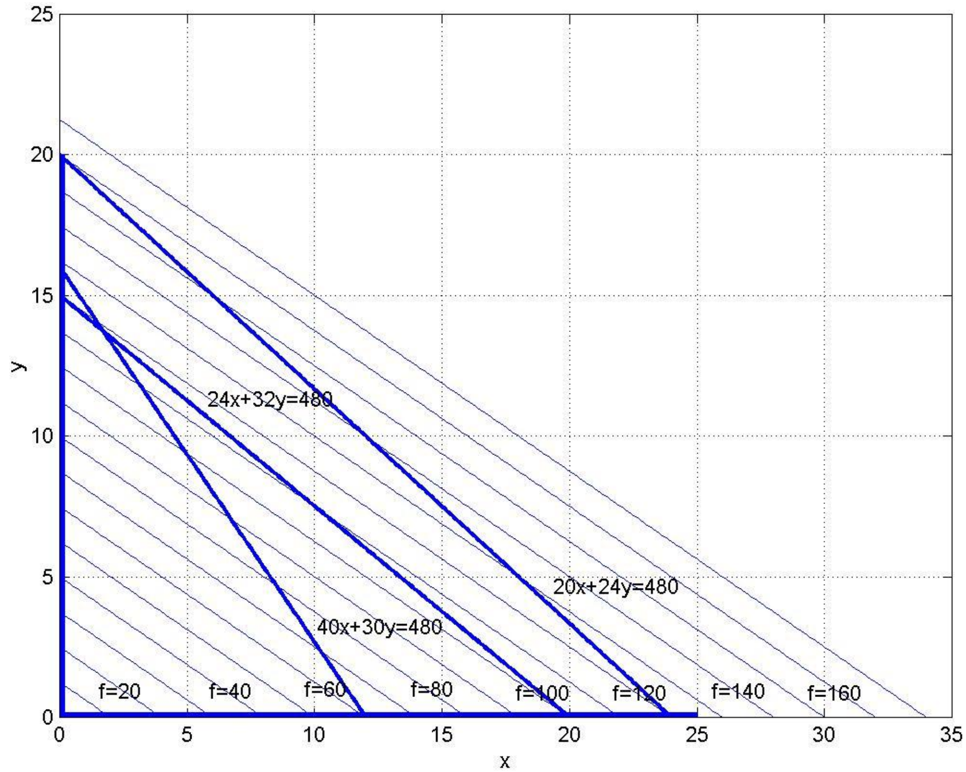

Fig. 4: Constraints with contours for Example 2

From Fig. 4, we observe that the *f*=120 contour appears to be the contour of maximum *f* which contains a point in the feasible region.

One should observe from Fig. 4 that the optimal point must occur at either the point (0,15) or the point of intersection between the two active constraints, which is (1.7143, 13.7143).

Because the slope of our contours are less than the slope of the constraint boundary 24x+32y=480, if a contour crosses the 24x+32y=480 constraint boundary within the feasible region, then there will necessarily be another point to the left of the crossing which will be *interior* (feasible, but not on a boundary) to the feasible region. This means it will be possible to move to a contour of higher *f* which still has at least one point in the feasible region.

By this argument, then, if a contour passes through the intersection point (1.7143, 13.7143), then it is possible to move to a contour of higher *f* which still has one point in the feasible region.

Because the optimal point must occur at either (0, 15) or (1.7143, 13.7143), we know now that it must occur at (0, 15).

Again, we notice: *the solution occurred at a point where two active constraints intersect.*

4.0 Example 3

Let's try a slightly different problem, as follows:

$$\max\ f(x,y) = 9x + 8y$$

Subject to

$$40x + 30y \le 480 \quad \text{(person 1)}$$

$$24x + 32y \le 480 \quad \text{(person 2)}$$

$$20x + 24y \le 480 \quad \text{(person 3)}$$

$$x \ge 0$$

$$y \ge 0$$

Figure 6 shows the feasible region together with contours of increasing *f*.

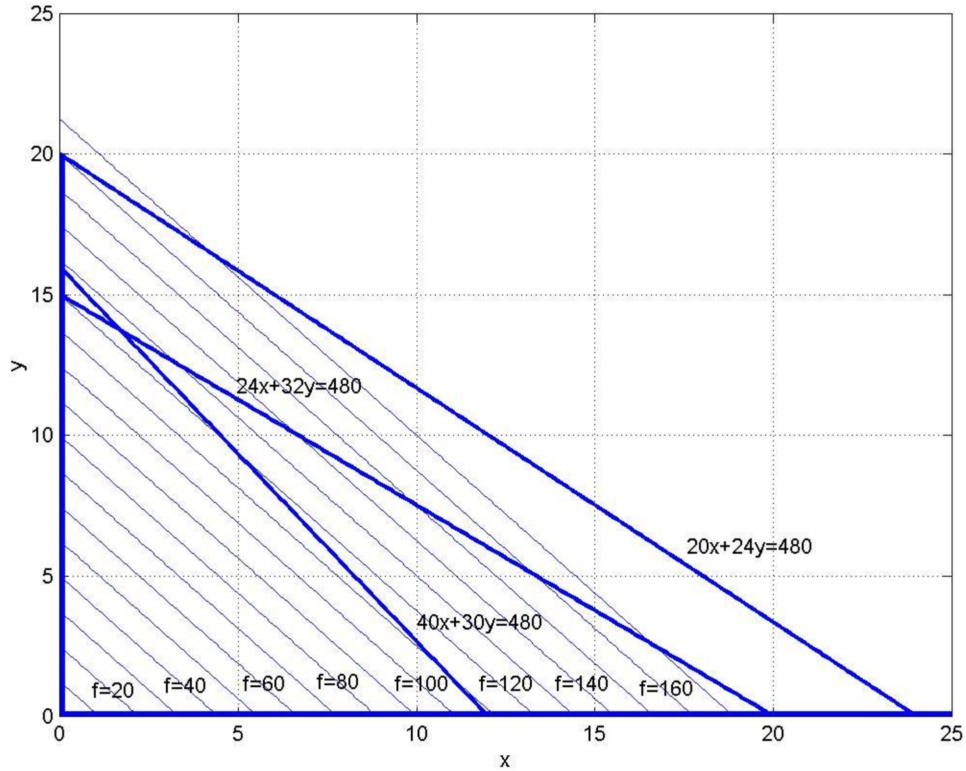

Fig. 6: Constraints with contours for Example 3

From Fig. 6, we see that the point (1.7143, 13.7143) must be the optimal point, and the maximum value of f will be about f=125 (evaluation of the above point yields f=125.14).

Again, we notice: *the solution occurred at a point where two active constraints intersect.*

<u>5.0 Example 4</u>

Let's again change our problem slightly, as follows:

$$\max\ f(x,y) = 15x + 8y$$

Subject to

$$40x + 30y \leq 480 \text{ (person 1)}$$

$$24x + 32y \leq 480 \text{ (person 2)}$$

$$20x + 24y \leq 480 \text{ (person 3)}$$

$$x \geq 0$$

$$y \geq 0$$

Figure 7 shows the feasible region together with contours of increasing *f*.

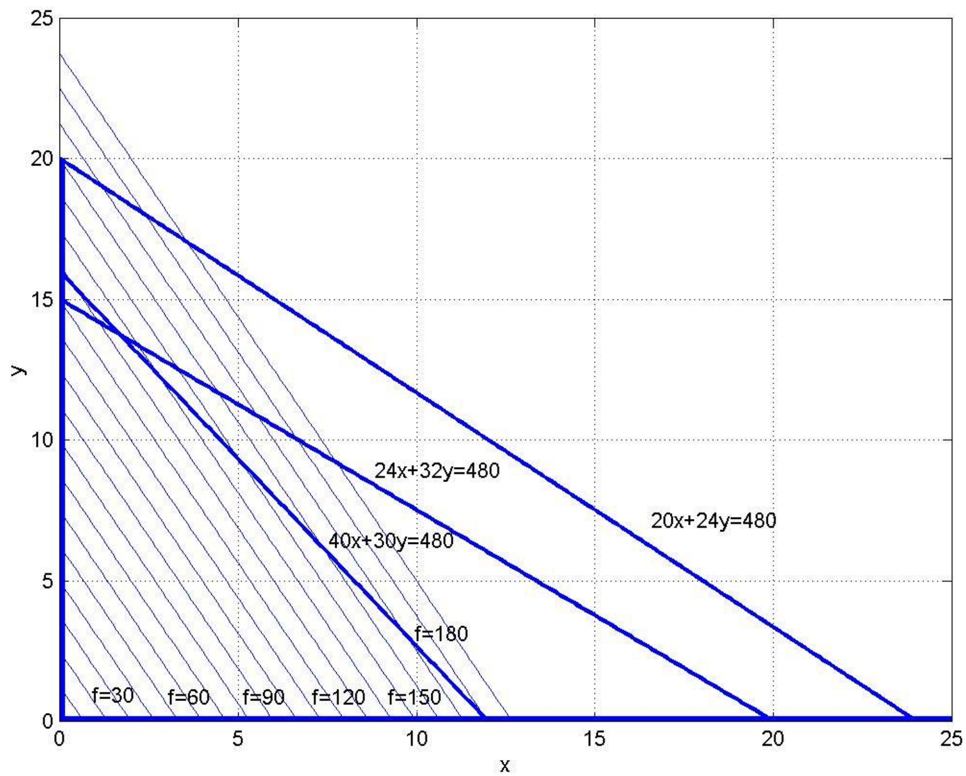

Fig. 7: Constraints with contours for Example 3

From Fig. 7, we see that the point (12, 0) must be the optimal point, and the maximum value of f will be f=180.

Again, as in all previous cases, we notice: *the solution occurred at a point where two active constraints intersect.*

This last observation is no coincidence. It always happens in a LP. That is:
The solution to an LP is always at a point where active constraints intersect.
We will call such intersection points *corner points*. Therefore we see that our solution will always be at a corner point. This provides us with a basis for solution to LPs: Search the corner points!

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[1] W. Claycombe and W. Sullivan, "Foundations of Mathematical Programming," Reston Publishing Company, Reston Virginia.