Duality #1

1.0 Second iteration for HW problem

Recall our LP example problem we have been working on, in equality form, is given below.

$$\max F = 3x_1 + 5x_2$$

s.t.

$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad\quad &= 4 \\ 2x_2 \quad\quad + x_4 \quad\quad &= 12 \\ 3x_1 + 2x_2 \quad\quad\quad\quad + x_5 &= 18 \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

which, when written in a slightly different form, is

$$\begin{aligned} F - 3x_1 - 5x_2 \quad\quad\quad\quad &= 0 \\ x_1 \quad\quad + x_3 \quad\quad\quad &= 4 \\ 2x_2 \quad\quad + x_4 \quad\quad &= 12 \\ 3x_1 + 2x_2 \quad\quad\quad\quad + x_5 &= 18 \end{aligned}$$

Recall that we performed the first iteration of the simplex method for it, which resulted in the following Tableau:

Tableau 1

Basic variable	Eq. #	Coefficients of						Right side	
		F	x_1	x_2	x_3	x_4	x_5		
F	0	1	-3	0	0	2.5	0	30	← Add 5 × pivot row
x_3	1	0	1	0	1	0	0	4	
x_2	2	0	0	1	0	0.5	0	6	
x_5	3	0	3	0	0	-1	1	6	← Add -2 × pivot row

Then when we tested for optimality, we discovered that we must do another iteration because the coefficient of x_1 is negative in the above tableau. I asked you to do this for homework. I will do it here, to provide you with the solution to the homework.

So our only choice for the entering variable is x_1. To determine the leaving variable, we identify the constraints that most limit the increase in x_1, as shown in Tableau 2 below.

Tableau 2

Basic variable	Eq. #	Coefficients of						Right side	
		F	x_1	x_2	x_3	x_4	x_5		
F	0	1	-3	0	0	2.5	0	30	
x_3	1	0	1	0	1	0	0	4	← $\frac{4}{1} = 4$
x_2	2	0	0	1	0	0.5	0	6	
x_5	3	0	3	0	0	-1	1	6	← $\frac{6}{3} = 2$

And so the constraint identifying the leaving variable is the last one, since its ratio is smallest (2<4).

The leaving variable is therefore x_5, since it is the variable in the last constraint which gets pushed to 0 as x_1 increases. And so the tableau is shown below with the pivot row, pivot column, and pivot element identified. We have also modified the last basic variable (column 1) to be x_1.

Tableau 3

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	-3	0	0	2.5	0	30
x_3	1	0	1	0	1	0	0	4
x_2	2	0	0	1	0	0.5	0	6
x_1	3	0	3	0	0	-1	1	6

Dividing the last equation by 3, we obtain:

Tableau 4

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	-3	0	0	2.5	0	30
x_3	1	0	1	0	1	0	0	4
x_2	2	0	0	1	0	0.5	0	6
x_1	3	0	1	0	0	-0.333	0.333	2

Using the last equation to eliminate the -3 in the top equation, we get:

Tableau 5

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	0	0	0	1.5	1	36
x_3	1	0	1	0	1	0	0	4
x_2	2	0	0	1	0	0.5	0	6
x_1	3	0	1	0	0	-0.333	0.333	2

Using the last equation to eliminate the 1 from the second equation, we get:

Tableau 6

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	0	0	0	1.5	1	36
x_3	1	0	0	0	1	0.333	-0.333	2
x_2	2	0	0	1	0	0.5	0	6
x_1	3	0	1	0	0	-0.333	0.333	2

The solution above is optimal because all coefficients in the objective function expression are positive.

2.0 Introduction to duality

Let's consider that our linear programming problem is actually a resource allocation problem where the various constraints are actually constraints on our resources. Recalling the original form of the problem, and giving it the name "Problem P":

Problem P

$$\begin{aligned} \max F &= 3x_1 + 5x_2 \\ \text{s.t. } x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned} \qquad (1)$$

we see that our new interpretation indicates that 4, 12, and 18 represent the maximum amount of each kind of resource that we have.

Now the question could very well arise: *how might we gain the most, in terms of the value of our optimized objective function, by increasing one resource or another*?

In order to provide a basis of comparison, let's allow each resource to increase by 1 unit. What will be the effect on the objective function?

To answer this question, I first solved the original LP, (1), in CPLEX. As we would expect, I got the answer F_1*=36, consistent with Tableau 6 above. (The subscript on *F*, which is "1" in this case, indicates it is the solution to the LP defined by (1) above).

Then I used CPLEX to solve LP (2).

$$\begin{aligned} \max F &= 3x_1 + 5x_2 \\ \text{s.t.}\quad x_1 \qquad &\le 5 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned} \tag{2}$$

where the only difference, relative to (1), is that the upper bound on the first constraint was increased from 4 to 5. CPLEX provides the answer F_2*=36. Apparently, increasing resources on this constraint has no effect.

Then I used CPLEX to solve LP (3):

$$\begin{aligned} \max F &= 3x_1 + 5x_2 \\ \text{s.t.} \quad x_1 \qquad &\leq 4 \\ 2x_2 &\leq 13 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned} \tag{3}$$

where the only difference, relative to (1), is that the upper bound on the second constraint was increased from 12 to 13. CPLEX provides the answer F_3*=37.5. Here, increasing the second resource by 1 unit provides that the objective function improves by an amount equal to 1.5.

Finally, I used CPLEX to solve LP (4):

$$\begin{aligned} \max F &= 3x_1 + 5x_2 \\ \text{s.t.} \quad x_1 \qquad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 19 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned} \tag{4}$$

where the only difference, relative to (1), is that the upper bound on the third constraint was increased from 18 to 19. CPLEX provides the answer F_4*=37. Here, increasing the third resource by 1 unit provides that the objective function improves by an amount equal to 1.

And so increasing the resources, i.e., the right hand sides of the first, second, and third constraints, by 1 unit, improve the optimal value of the objective function by 0, 1.5, and 1, respectively. In other words,

$$\frac{\Delta F^*}{\Delta b_1} = 0$$
$$\frac{\Delta F^*}{\Delta b_2} = 1.5 \qquad (5)$$
$$\frac{\Delta F^*}{\Delta b_3} = 1$$

This is quite useful information, in that it could guide our future allocation (or reallocation) or resources. In other words, assume *F** represents my optimal profits, and I find I have a little extra money to spend. Then (5) tells me it is better to spend that money to increase resource 2 than to increase resource 3, and that it will do me no good at all to increase resource 1. It is of interest to inspect Tableau 6 at this point.

Tableau 6

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	0	0	0	1.5	1	36
x_3	1	0	0	0	1	0.333	-0.333	2
x_2	2	0	0	1	0	0.5	0	6
x_1	3	0	1	0	0	-0.333	0.333	2

Notice one very interesting thing:

The coefficients of the slack variables in this final tableau are exactly the same as the right-hand-sides of (5): 0, 1.5, and 1. These values are circled in Tableau 6.

This is no coincidence. In fact, it will always happen. That is:

> *The coefficients of the slack variables in the objective function expression of the final tableau give the improvement in the objective for a unit increase in the right-hand-sides of the corresponding constraints.*

It is of interest to examine the units of these coefficients, an exercise most effectively accomplished by returning to (5) where we can see that they have units of (units of *F*)/(units of *b*). For example, if *F* is measured in dollars, and *b* is measured in, say, pipe fittings, then these coefficients would have units of $/pipe fitting.

These slack variable coefficients have names. They are called the *dual variables*. We will see why they are called dual variables later. They are also called *shadow prices*. For now, let's give them the nomenclature λ_i corresponding to the i^{th} constraint.

One note of caution, here. Recalling that λ_3=1 in our problem, we understand that if we increase b_3 by 1, from 18 to 19, we will improve our objective function value (at the optimum) by 1, from 36 to 37, and that is indeed the case.

But what if we increase b_3 by 6, making it 24? Will we see an increase in F^* to 42? Use of CPLEX indicates this **does** turn out to be the case.

However, if we increase b_3 to 26, we still obtain F^*=42, a result which suggests that somewhere between b_3=24 and b_3=26, the third constraint became no longer binding (it was no longer one of the two constraints that defined the corner point).

This is a result of the fact that whenever one has multiple resources, each of which is constrained, it will be the case that only a subset of the resources actually limit the objective. For example, you may have labor hours, trucks, & pipe fittings as your resources, each of which are individually constrained. You have a large truck fleet and a whole warehouse of pipe fittings, but you don't have enough labor. And so you increase labor until you hit your limit on pipe fitting inventory. You then begin increasing pipe fitting inventory, but soon you hit the limit on trucks.

3.0 Motivating the dual problem

Consider again our original problem.

$$
\begin{aligned}
\max F &= 3x_1 + 5x_2 \\
\text{s.t.}\quad x_1 \quad\quad &\le 4 \\
2x_2 &\le 12 \\
3x_1 + 2x_2 &\le 18 \\
x_1 \ge 0,\ x_2 &\ge 0
\end{aligned}
\qquad (6)
$$

Let's express linear combinations of multiples of the constraints, where the multipliers on the constraints are denoted λ_i on the i^{th} constraint. The below problem illustrates (the λ_i must be nonnegative):

$$
\begin{aligned}
&\max F = 3x_1 + 5x_2 \\
&\text{s.t.}\quad (x_1 \qquad\quad \le 4)\lambda_1 \\
&\quad + \qquad (2x_2 \le 12)\lambda_2 \\
&\quad + (3x_1 + 2x_2 \le 18)\lambda_3 \qquad (7) \\
\hline
&(x_1\lambda_1 + 3x_1\lambda_3) + (2x_2\lambda_2 + 2x_2\lambda_3) \le 4\lambda_1 + 12\lambda_2 + 18\lambda_3
\end{aligned}
$$

The last relation can be rewritten as follows:

$$
\begin{aligned}
&\max F = 3x_1 + 5x_2 \\
&\text{s.t.}\quad (x_1 \qquad\quad \le 4)\lambda_1 \\
&\quad + \qquad (2x_2 \le 12)\lambda_2 \\
&\quad + (3x_1 + 2x_2 \le 18)\lambda_3 \qquad (8) \\
\hline
&(\lambda_1 + 3\lambda_3)x_1 + (2\lambda_2 + 2\lambda_3)x_2 \le 4\lambda_1 + 12\lambda_2 + 18\lambda_3
\end{aligned}
$$

"COMPOSITE INEQUALITY"

The left-hand-side of the composite inequality, being a linear combination of our original inequalities, **must hold** at any feasible solution (x_1,x_2), and in particular, at the optimal solution (x_1^*, x_2^*). That is, it is a necessary condition for satisfying the inequalities from which it came[1].

Write the objective function expression together with the composite inequality:

$$F = 3x_1 + 5x_2$$

$$(\lambda_1 + 3\lambda_3)x_1 + (2\lambda_2 + 2\lambda_3)x_2 \le 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \quad (9)$$

Let's develop criteria for selecting λ_1, λ_2, λ_3.

Consider the following 5 concepts…*watch closely*:

Concept 1: Make sure that our choices of λ_1, λ_2, λ_3 are such that each coefficient of x_i in the composite inequality is at least as great as the corresponding coefficient in the objective function expression, i.e.,

$$\begin{aligned} \lambda_1 + 3\lambda_3 &\ge 3 \\ 2\lambda_2 + 2\lambda_3 &\ge 5 \end{aligned} \quad (10)$$

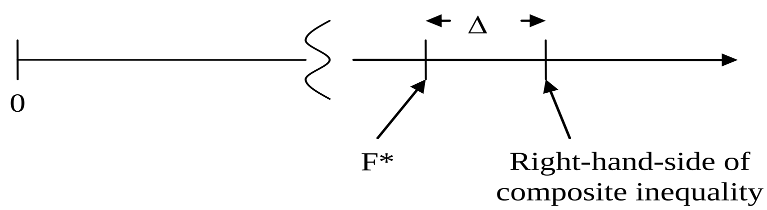
This guarantees that any solution (x_1,x_2) results in a value F which is less than or equal to the left-hand-side of the composite inequality.

---

[1] Although necessary, it is not sufficient. A simple example will show this. Choose $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and the combined inequality is then $4x_1 + 4x_2 \le 34$. All values of (x_1, x_2) that satisfy the original inequalities must satisfy this one, but there will be some that satisfy this one that do not satisfy one or more of the original inequalities, for example, (4,4) results here in $32 \le 34$, but the third inequality of the original ones results in 3(4)+2(4)=20 which is greater than its right-hand-side of 18.

Concept 2: Because the left-hand side of the composite inequality is less than or equal to the right hand side of the composite inequality, we can also say that any solution (x_1,x_2) results in a value *F* which is less than or equal to the right-hand-side of the composite inequality.

Concept 3: Concept 2 implies the right-hand-side of the composite inequality is an upper bound on the value that *F* may take. This is true for any value of *F*, even the maximum value *F**. In other words, if we look at the value *F** and the right-hand-side of the composite inequality on the real number line, they appear as below, with *F** to the left, and there would be some difference Δ between them.



Concept 4: Now choose λ_1, λ_2, λ_3 to *minimize* the right-hand-side of the composite inequality, subject to constraints (10). This creates a least upper bound to *F**, i.e., it pushes the right-hand-side of the composite inequality as far left as possible, while guaranteeing right-hand-side remains greater than or equal to *F** (due to enforcement of constraints (10)).

Concept 5: Given that the right-hand-side of the composite inequality is an upper bound to *F**, then finding its minimum, subject to (10), implies Δ=0.

Concept 5 tells us that if we solve the following problem, call it Problem D,

Problem D

$$\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3$$

subject to

$$\lambda_1 + 3\lambda_3 \geq 3$$

$$2\lambda_2 + 2\lambda_3 \geq 5$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$$

that the value of the obtained objective function, at the optimum, will be the same as the value of the original objective function at its optimum, i.e., $F^*=G^*$.

In other words, solving Problem D is equivalent to solving Problem P.

$$\underbrace{\begin{aligned} &\underline{\text{Problem P}} \\ &\max F = 3x_1 + 5x_2 \\ &\text{s.t.}\ \ x_1 \leq 4 \\ &\qquad 2x_2 \leq 12 \\ &\qquad 3x_1 + 2x_2 \leq 18 \\ &\qquad x_1 \geq 0, x_2 \geq 0 \end{aligned}}_{\text{Primal Problem}} \quad \Leftrightarrow \quad \underbrace{\begin{aligned} &\underline{\text{Problem D}} \\ &\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ &\text{subject to} \\ &\lambda_1 + 3\lambda_3 \geq 3 \\ &2\lambda_2 + 2\lambda_3 \geq 5 \\ &\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0 \end{aligned}}_{\text{Dual Problem}}$$

Problem P is called the primal problem. Problem D is called the dual problem. They are precisely equivalent. To show this, let's use CPLEX to solve Problem D.

The CPLEX code to do it is below (note I am using *y* instead of λ):

```
minimize
 1 y1 + 12 y2 + 18 y3
subject to
 1 y1 + 3 y3 >= 3
 2 y2 + 2 y3 >=5
 y1 >= 0
 y2 >= 0
 y3 >= 0
end
```

The solution gives

$G^*=36$

$\lambda_1=0$

$\lambda_2=1.5$

$\lambda_3=1$

It is of interest to inspect Tableau 6 of the primal.

Tableau 6

Basic variable	Eq. #	Coefficients of						Right side
		F	x_1	x_2	x_3	x_4	x_5	
F	0	1	0	0	0	1.5	1	36
x_3	1	0	0	0	1	0.333	-0.333	2
x_2	2	0	0	1	0	0.5	0	6
x_1	3	0	1	0	0	-0.333	0.333	2

We note the values of the decision variables obtained for the dual problem are exactly the coefficients of the slack variables in the primal problem.

We also note that $G^*=F^*=36$.

One last thing that is very interesting here. Using CPLEX, following solution of the dual, we can also obtain the coefficients for the dual problem slack variables (the slack variables to problem D), in the objective function row, using the following command,

```
display solution dual -
```

This command was made after solution of the above "dual" problem in CPLEX, therefore it is giving the solution to the "dual of the dual," which is the primal!

and they are:

λ_4 → 2.0

λ_5 → 6.0

which is precisely the solution of the primal problem, $x_1=2$, $x_2=6$, as can be read off from Tableau 6 above.

Caution to avoid confusion: The above values for λ_4 and λ_5 are **not** the *values* of the slack variables. They are the *coefficients* of the slack variables of the objective function expression in the last tableau of the dual problem.

This suggests that there is a certain circular relationship here, which can be stated as

The dual of the dual to a primal is the primal.

That is, if you called Problem D our primal problem, and took its dual, you would get our original primal problem back, as illustrated below.

Problem D

$$\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3$$

subject to

$$\underbrace{\begin{aligned} &\lambda_1 + 3\lambda_3 \ge 3 \\ &2\lambda_2 + 2\lambda_3 \ge 5 \\ &\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{aligned}}_{\textit{Primal Problem}}$$

$\Leftrightarrow$

Problem P

$$\max F = 3x_1 + 5x_2$$

$$\underbrace{\begin{aligned} \text{s.t.}\quad x_1 \qquad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned}}_{\text{Dual Problem}}$$

4.0 Obtaining the dual from the primal

It is useful to make the following observations:

1. Number of decision variables and constraints:
   - Number of dual decision variables is number of primal constraints.
   - Number of dual constraints is number of primal decision variables.
2. Coefficients of decision variables in dual objective are right-hand-sides of primal constraints.

Problem P

$$\max F = 3x_1 + 5x_2$$

$$\underbrace{\begin{aligned} \text{s.t.}\quad x_1 \qquad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned}}_{\text{Primal Problem}}$$

$\Leftrightarrow$

Problem D

$$\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3$$

subject to

$$\underbrace{\begin{aligned} &\lambda_1 + 3\lambda_3 \ge 3 \\ &2\lambda_2 + 2\lambda_3 \ge 5 \\ &\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{aligned}}_{\text{Dual Problem}}$$

3. Coefficients of decision variables in primal objective are right-hand-sides of dual constraints.

$$\underbrace{\begin{aligned} &\underline{\text{Problem P}} \\ &\max F = 3x_1 + 5x_2 \\ &\text{s.t. } x_1 \le 4 \\ &2x_2 \le 12 \\ &3x_1 + 2x_2 \le 18 \\ &x_1 \ge 0, x_2 \ge 0 \end{aligned}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{aligned} &\underline{\text{Problem D}} \\ &\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ &\text{subject to} \\ &\lambda_1 + 3\lambda_3 \ge 3 \\ &2\lambda_2 + 2\lambda_3 \ge 5 \\ &\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{aligned}}_{\text{Dual Problem}}$$

4. Coefficients of one variable across multiple primal constraints are coefficients of multiple variables in one dual constraint.

$$\underbrace{\begin{aligned} &\underline{\text{Problem P}} \\ &\max F = 3x_1 + 5x_2 \\ &\text{s.t. } x_1 \le 4 \\ &2x_2 \le 12 \\ &3x_1 + 2x_2 \le 18 \\ &x_1 \ge 0, x_2 \ge 0 \end{aligned}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{aligned} &\underline{\textit{Problem D}} \\ &\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ &\textit{subject to} \\ &\lambda_1 + 3\lambda_3 \ge 3 \\ &2\lambda_2 + 2\lambda_3 \ge 5 \\ &\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{aligned}}_{\textit{Dual Problem}}$$

$$\underbrace{\begin{aligned} &\underline{\text{Problem P}} \\ &\max F = 3x_1 + 5x_2 \\ &\text{s.t. } x_1 \le 4 \\ &2x_2 \le 12 \\ &3x_1 + 2x_2 \le 18 \\ &x_1 \ge 0, x_2 \ge 0 \end{aligned}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{aligned} &\underline{\textit{Problem D}} \\ &\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ &\textit{subject to} \\ &\lambda_1 + 3\lambda_3 \ge 3 \\ &2\lambda_2 + 2\lambda_3 \ge 5 \\ &\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{aligned}}_{\textit{Dual Problem}}$$

5. Coefficients of one variable across multiple dual constraints are coefficients of multiple variables in one primal constraint.

$$\underbrace{\begin{aligned}&\underline{\text{Problem P}}\\&\max F = 3x_1 + 5x_2\\&\text{s.t. } x_1 \le 4\\&2x_2 \le 12\\&3x_1 + 2x_2 \le 18\\&x_1 \ge 0, x_2 \ge 0\end{aligned}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{aligned}&\underline{\textit{Problem D}}\\&\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3\\&\textit{subject to}\\&\lambda_1 + 3\lambda_3 \ge 3\\&2\lambda_2 + 2\lambda_3 \ge 5\\&\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0\end{aligned}}_{\textit{Dual Problem}}$$

$$\underbrace{\begin{aligned}&\underline{\text{Problem P}}\\&\max F = 3x_1 + 5x_2\\&\text{s.t. } x_1 \le 4\\&2x_2 \le 12\\&3x_1 + 2x_2 \le 18\\&x_1 \ge 0, x_2 \ge 0\end{aligned}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{aligned}&\underline{\textit{Problem D}}\\&\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3\\&\textit{subject to}\\&\lambda_1 + 3\lambda_3 \ge 3\\&2\lambda_2 + 2\lambda_3 \ge 5\\&\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0\end{aligned}}_{\textit{Dual Problem}}$$

$$\underbrace{\begin{aligned}&\underline{\text{Problem P}}\\&\max F = 3x_1 + 5x_2\\&\text{s.t. } x_1 \le 4\\&2x_2 \le 12\\&3x_1 + 2x_2 \le 18\\&x_1 \ge 0, x_2 \ge 0\end{aligned}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{aligned}&\underline{\textit{Problem D}}\\&\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3\\&\textit{subject to}\\&\lambda_1 + 3\lambda_3 \ge 3\\&2\lambda_2 + 2\lambda_3 \ge 5\\&\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0\end{aligned}}_{\textit{Dual Problem}}$$

6. If primal objective is maximization, then dual objective is minimization.
7. If primal constraints are ≤, dual constraints are ≥.

From the above, we should be able to immediately write down the dual given the primal.

Example:

Let's return to the example we used to illustrate use of CPLEX in the notes called "Intro_CPLEX."

$$\max F = 5x_1 + 4x_2 + 3x_3$$

Subject to

$$2x_1 + 3x_2 + x_3 \le 5$$

$$4x_1 + x_2 + 2x_3 \le 11$$

$$3x_1 + 4x_2 + 2x_3 \le 8$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$$

The dual problem can be written down by inspection.

$$\min G = 5\lambda_1 + 11\lambda_2 + 8\lambda_3$$

Subject to

$$2\lambda_1 + 4\lambda_2 + 3\lambda_3 \ge 5$$

$$3\lambda_1 + \lambda_2 + 4\lambda_3 \ge 4$$

$$\lambda_1 + 2\lambda_2 + 2\lambda_3 \ge 3$$

$$\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0$$

We can use CPLEX to check. First, we solve the primal problem using the below code:

```
maximize
 5 x1 + 4 x2 + 3 x3
subject to
 2 x1 + 3 x2 + x3 <= 5
 4 x1 + x2 + 2x3 <= 11
 3 x1 + 4 x2 + 2 x3 <= 8
 x1 >= 0
 x2 >= 0
 x3 >= 0
end
```

The solution is $(x_1,x_2,x_3)=(2,0,1)$, $F^*=13$.
The dual variables are: $(\lambda_1, \lambda_2, \lambda_3)=(1,0,1)$

Now, solve the dual problem using the below code:

```
minimize
 5 y1 + 11 y2 + 8 y3
subject to
 2 y1 + 4 y2 + 3 y3 >= 5
 3 y1 + 1 y2 + 4 y3 >= 4
 1 y1 + 2 y2 + 2 y3 >= 3
 y1 >= 0
 y2 >= 0
 y3 >= 0
end
```

The solution is $(\lambda_1, \lambda_2, \lambda_3)=(1,0,1)$, $G^*=13$.
The dual variables are: $(x_1,x_2,x_3)=(2,0,1)$.

5.0 Viewing the primal-dual relationship

Another way to view the relationship between the primal and the dual is via use of the primal-dual table. Although this offers no new information relative to what we have already learned, you might find it helpful in remembering the structural aspects to the relationship.

Let's consider the following generalized primal-dual problems.

$$\begin{aligned} &\max F = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ &s.t. \\ &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ &\;\vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\ &x_1, x_2, \ldots, x_n \ge 0 \end{aligned}$$

$$\begin{aligned} &\min G = b_1\lambda_1 + b_2\lambda_2 + \ldots + b_m\lambda_m \\ &s.t. \\ &a_{11}\lambda_1 + a_{21}\lambda_2 + \ldots + a_{m1}\lambda_m \ge c_1 \\ &a_{12}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{m2}\lambda_m \ge c_2 \\ &\;\vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ &a_{1n}\lambda_1 + a_{2n}\lambda_2 + \ldots + a_{mn}\lambda_m \ge c_n \\ &\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0 \end{aligned}$$

The primal-dual table is shown below.

			Primal Problem					
			Coefficients of				Right side	
			x_1	x_2	…	x_n		
Dual Problem	Coefficients of	λ_1	a_{11}	a_{12}	…	a_{1n}	$\leq b_1$	Coefficients for dual objective function
		λ_2	a_{21}	a_{22}	…	a_{2n}	$\leq b_2$	
		…	⋮	⋮	⋮	⋮	⋮	
		λ_m	a_{m1}	a_{m2}	…	a_{mn}	$\leq b_m$	
	Right side		$\geq$	$\geq$		$\geq$		
			c_1	c_2	…	c_n		
			Coefficients for primal objective function					

For example, our previous example problem has a primal-dual table as shown below.

			Primal Problem				
			Coefficients of			Right side	
			x_1	x_2	x_3		
Dual Problem	Coefficients of	λ_1	2	3	1	≤ 5	Coefficients for dual objective function
		λ_2	4	1	2	≤ 11	
		λ_3	3	4	2	≤ 8	
	Right side		$\geq$	$\geq$	$\geq$		
			5	4	3		
			Coefficients for primal objective function				

6.0 The duality theorem

We have already been using this theorem, and so now we merely formalize it….

Duality theorem: If the primal problem has an optimal solution $\underline{x}^*$, then the dual problem has an optimal solution $\underline{\lambda}^*$ such that

$$G(\underline{\lambda}^*) = F(\underline{x}^*)$$

The proof is given in [1, pp. 58-59].

The duality theorem raises an interesting question. What if the primal does not have an optimal solution? Then what happens in the dual?

To answer this, we must first consider what are the alternatives for finding an optimal solution to the primal? There are two:

1. The primal is unbounded.
2. The primal is infeasible.

7.0 Unbounded primal

We have already seen an example of an unbounded primal, illustrated by the problem below and its corresponding feasible region.

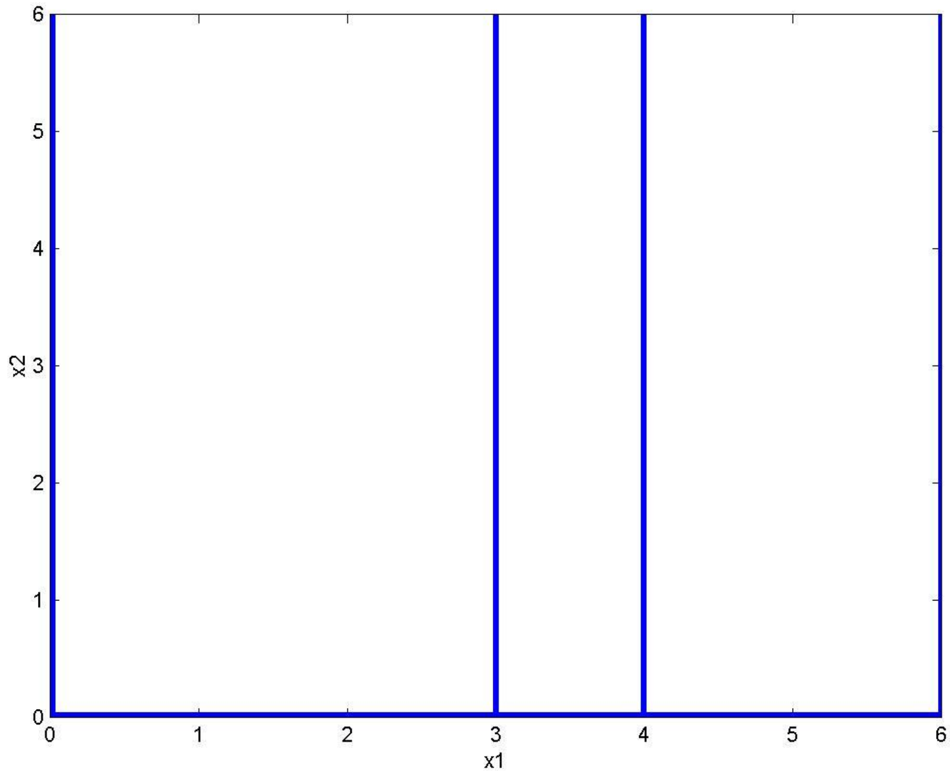
$$\max F = 3x_1 + 5x_2$$

$$\text{s.t.} \quad x_1 \le 4$$

$$4x_1 \le 12$$

$$3x_1 \le 18$$

$$x_1 \ge 0, x_2 \ge 0$$



Recall that the objective function for the dual establishes an upper bound for the objective function of the primal, i.e.,

$$G(\underline{\lambda}) \geq F(\underline{x})$$

for any sets of feasible solutions $\underline{\lambda}$ and $\underline{x}$.
If $F(\underline{x})$ is unbounded, then the only possibility for $G(\underline{\lambda})$ is that it must be infeasible.

Let's write down the dual of the above primal to see.

$$\min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3$$
$$s.t.$$
$$\lambda_1 + 4\lambda_2 + 3\lambda_3 \geq 3$$
$$0\lambda_1 + 0\lambda_2 + 0\lambda_3 \geq 5$$
$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$$

and we immediately see that the second constraint cannot be satisfied, and so the dual is infeasible.
Likewise, we can show that if the dual is unbounded, the primal must be infeasible.
However, it is not necessarily true that if the primal (or dual) is infeasible, that the dual (or primal) is unbounded. It is possible for an infeasible primal to have an infeasible dual and vice-versa, that is, both the primal and the dual may be both be infeasible. Reference [1, p. 60] provides such a case.

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[1] V. Chvatal, "Linear Programming," Freeman & Company, NY, 1983.