Linearized optimal power flow

1.0 Some introductory comments

We desire to bring in the network constraints into the problem in order to study the influence of transportation (transmission) constraints on the most economic distribution of generation. To maintain simplicity, we will assume that each generator makes only a single-price offer. Thus, each unit is represented in the objective function by a constant times the generation level for that unit. So here is the formal statement of our problem:

$$\min \sum_{k\in\{generator\_buses\}} s_k P_{gk}$$ [1]

Subject to:

$$\underline{P} = \underline{B}'\,\underline{\theta}$$ [2]

$$\underline{P}_B = (\underline{D}\times\underline{A})\times\underline{\theta}$$ [3]

$$-\underline{P}_{B,\max} \le \underline{P}_B \le \underline{P}_{B,\max}$$ [4]

$$0 \le P_{gk} \le P_{gk,\max}, \forall k \in \{generator\_buses\}$$ [5]

where

$$P_k = P_{gk} - P_{dk}, k = 1,\ldots N$$ [6]

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[1] k is index on the units.

[2] These are DC power flow equations to represent the network. However, we must include all nodal injections $P_1,\ldots P_N$ and all angles $\theta_1 \ldots \theta_N$ in this set of equations.

[3] These are equation to get line flows. Again, we need to include all angles $\theta_1 \ldots \theta_N$ in this set of equations.

[4] These are the limits on the line flows. Notice that there is only one circuit rating, but it must be enforced as a limit if the flow is in one direction or in the other.

[5] These are the limits on the linear cost curve variables.

[6] This relates variables used in the cost curves (P_{kj}) to variables used in DC power flow equations (P_k).

Before we proceed with formulating this problem, we must make one very important observation. In the DC power flow problem, we found that if we included an equation for all buses in the network, using all angles in the network, that there was a dependency in our set of equations, and the matrix was not invertable.

Now, however, we desire to include all N equations for our network because we want a Lagrange multiplier (corresponding to the LMP) for each bus. This implies that we must also include all of the angles as our unknowns. This will not create the same problem of matrix singularity that we had before because our overall system of equations will include (2), (3), and (6). In fact, this system will be under-constrained (there will be more unknowns than equations). This is acceptable because

- we are not trying to solve them (because they are underconstrained, there are many solutions);
- rather, we are trying to minimize a function that is subject to them.(with appropriate convexity requirements, there is only one solution that will do this).

Therefore we will include in $\underline{P}=\underline{B}'\underline{\theta}$ all DC power flow equations, one for each node, as a function of all angles in the network.

So be aware that in all of what follows, the vector $\underline{P}$ includes the reference bus injection, the vector $\underline{\theta}$ includes the reference bus angle, the matrix $\underline{B}'$ is N×N (i.e., it does not have a row and column eliminated corresponding to the reference bus), and adjacency (node-arc) matrix $\underline{D}$ is M×N (i.e, we do not eliminate the column corresponding to the reference bus).

To solve this problem as a linear program, we need to be able to write it in a form that a standard LP solver will handle. This means equality and inequality constraints must be written as a function of a single vector of unknowns (the "solution vector").

<u>Special note on LP solver</u>: The below formulation was developed for use in Matlab. However, this formulation may also be implemented in CPLEX. To do so, you need to realize that CPLEX is quite flexible in handling input constraints. The below example illustrates this flexibility:

```
\Problem name: alldiet.lp

Minimize
 DOLLARS: 10.89 pizza + 0.79 FRFries + 2.89 M.cherry + 2.59 MILK + 2.69 C.Milk
 + 0.47 P.towels + 1.49 !SUGARS! + 1.27 cereal + 3.29 SixPackX + 3.29 SixPackY
Subject To
 aluminum: 137 pizza + 71.4 SixPackX + 71.4 SixPackY >= 27230
 Vacuous:  >= 0
 RED#2:    1.9 pizza + 18 M.cherry + 1.9 P.towels + 1.6 !SUGARS! + 1.6 cereal
           + 1.7 SixPackX + 1.7 SixPackY - RgRED#2  = 18390
 salt:     67.2 pizza + 36 FRFries + 2.6 M.cherry + 3.4 P.towels + 4.3 !SUGARS!
           + 4.3 cereal + 31.4 SixPackX + 31.4 SixPackY + 0.000067 Air >= 21740
 fat:      41.8 pizza + 210.6 FRFries + 243 MILK + 223 C.Milk + 1.3 SixPackX
           + 1.2 SixPackY - Rgfat  = 30270
 Fiber:    3.4 pizza + 9.3 FRFries + 0.084 M.cherry + 0.45 P.towels
           + 1.78 !SUGARS! + 1.78 cereal - RgFiber  = 4789
 CALCIUM:  45.2 MILK + 43.2 C.Milk - RgCALCIUM  = 11460
 Sparkle:  - pizza - FRFries <= 10000
 Dirt:     pizza + FRFries >= -10000
Bounds
      FRFries >= 20
 500 <= M.cherry <= 800
 20 <= MILK <= 5000
 0 <= C.Milk <= 8888

       P.towels >= -27000
       !SUGARS! >= 144
  0 <= SixPackX <= 99999999
       SixPackY = 24
       Air Free
 -10 <= RgRED#2 <= 0
  0 <= Rgfat <= 99999
 -34789 <= RgFiber <= 0
  0 <= RgCALCIUM <= 240
 End
```

Equality constraints

Less-than inequality constraint

Greater-than inequality constraint

Less-than, greater-than inequality constraints

You will find a great deal of additional information on how to use CPLEX at www.iro.umontreal.ca/~gendron/IFT6551/CPLEX/HTML.

2.0 Formulation of the linearized OPF

We define the following solution vector for our problem as:

$$\underline{x} = \begin{bmatrix} \underline{P}_g \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} \qquad (1)$$

where

- $\underline{P}_g$ is the vector of generation increments $[P_{gk} \ \ldots]^T$ for all bus k that has generation.
- $\underline{P}_B$ is the vector of line flows $[P_{b1} \ P_{b2} \ldots P_{bM}]^T$, M is # of branches.
- $\underline{\theta}$ is the vector of bus angles, in radians $[\theta_1 \quad \theta_2 \ldots \theta_N]$, N is # of buses.

We want to capture all of our equality constraints in a single matrix relation.

There are two kinds of equality constraints: one due to line flows

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \qquad (2)$$

and the other due to injections:

$$\underline{P} = \underline{B}' \underline{\theta} \qquad (3)$$

We rewrite these slightly modified to make them more convenient to embed in a single matrix equation.

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \qquad (4)$$

$$-\underline{P} + \underline{B}' \underline{\theta} = \underline{0} \qquad (5)$$

The inequality constraints are straightforward given our definition of the solution vector. The only issue here is what to use as constraints on the angles? Clearly, all angles must reside between –π radians and +π radians. Therefore, the inequality constraints will be:

$$\begin{bmatrix} \underline{P}_{g,\min} \\ -\underline{P}_{B,\max} \\ -\underline{\pi} \end{bmatrix} \le \begin{bmatrix} \underline{P}_{g} \\ \underline{P}_{B} \\ \underline{\theta} \end{bmatrix} \le \begin{bmatrix} \underline{P}_{g} \\ \underline{P}_{B,\max} \\ \underline{\pi} \end{bmatrix} \qquad (6)$$

Example:

We illustrate using an example that utilizes the same system we have been using in our previous notes, where we had 3 units connected to 3 different buses in a 4 bus network supplying load at 2 different buses.

The one-line diagram for the example system is given in Fig. 1. We will modify the load so that it has a total of 2.1787 per unit (or 217.87 MW), with 1 per unit load at bus 2 (P_{d2}=1.0) and 1.1787 per unit load at bus 3 (P_{d3}=1.1787).

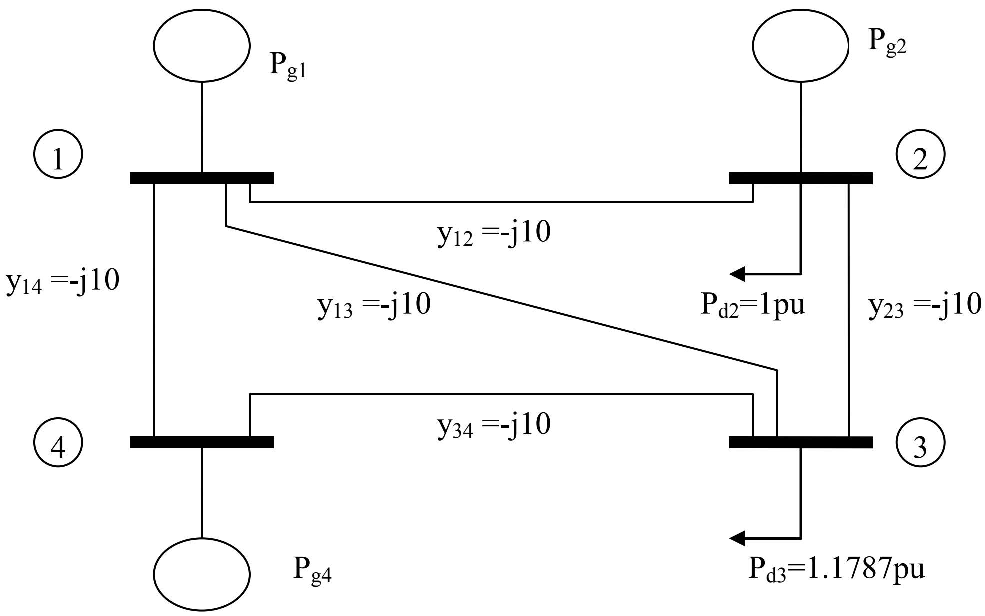

Fig. 1: One line diagram for example system

The offers and corresponding min and max generation limits are as follows:

$$K_1(P_{g1}) = s_1 P_{g1},\ 50 \le P_{g1} \le 200$$

$$K_2(P_{g2}) = s_2 P_{g2},\ 37.5 \le P_{g2} \le 150$$

$$K_4(P_{g4}) = s_4 P_{g4},\ 45 \le P_{g4} \le 180$$

with

s_1=13.07 $/MWhr
s_2=12.11 $/MWhr
s_4=12.54 $/MWhr

A modification is necessary at this point in that we need to **change the generation variables to per-unit.** This is necessary in order to bring in the transmission equations, which are in per-unit. Thus, the decision variables P_{g1}, P_{g2}, and P_{g3}, are all divided by 100. We can compensate for this in terms of obtaining the correct evaluation for the cost by multiplying the slopes by 100. Thus, we will use:

s_1=1307 $/puMWhr
s_2=1211 $/puMWhr
s_4=1254 $/puMWhr

Objective function: Let's explicitly write out the solution vector, because then we will be able to write out the objective function immediately.

$$\underline{x} = \begin{bmatrix} \underline{P}_g \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

Now we see that the objective function is given by:

$$\boldsymbol{Z(\underline{x})} = \begin{bmatrix} \mathbf{1307} & \mathbf{1211} & \mathbf{1254} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{P_{g1}} \\ \boldsymbol{P_{g2}} \\ \boldsymbol{P_{g4}} \\ \boldsymbol{P_{B1}} \\ \boldsymbol{P_{B2}} \\ \boldsymbol{P_{B3}} \\ \boldsymbol{P_{B4}} \\ \boldsymbol{P_{B5}} \\ \boldsymbol{\theta_1} \\ \boldsymbol{\theta_2} \\ \boldsymbol{\theta_3} \\ \boldsymbol{\theta_4} \end{bmatrix}$$

Equality constraints: The equality constraints are given in eqs. (4) and (5), repeated here for convenience:

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \qquad (4)$$

$$-\underline{P} + \underline{B}'\underline{\theta} = \underline{0} \qquad (5)$$

We will build all of these equality constraints into a matrix form of $\underline{A}_{eq}\underline{x}=\underline{b}_{eq}$. We begin by noting dimensions.

- Columns: Since the solution vector $\underline{x}$ is 12×1, $\underline{A}_{eq}$ must have 12 columns in order to pre-multiply $\underline{x}$.
- Rows: Since there are 5 branches, eq. (4) will contribute 5 rows to $\underline{A}_{eq}$. Since there are 4 buses, eq. (5) will contribute 4 rows to $\underline{A}_{eq}$. So $\underline{A}_{eq}$ will have total of 9 rows.

Therefore, the dimensions of $\underline{A}_{eq}$ will be 9×12.

We begin with the line flow equations, eq. (4). From the notes on "Power flow equations," we can recall the D and A matrices. The D matrix is exactly the same as before, which is:

$$\underline{D}=\begin{bmatrix}10&0&0&0&0\\0&10&0&0&0\\0&0&10&0&0\\0&0&0&10&0\\0&0&0&0&10\end{bmatrix}$$

But the node-arc incidence matrix, A, must be modified to account for the fact that we now have 4 angle variables. So it will get another column to multiply the new variable, which is θ_1:

$$\underline{A}=\begin{bmatrix}1&0&0&-1\\1&-1&0&0\\0&1&-1&0\\0&0&-1&1\\1&0&-1&0\end{bmatrix}$$

The D×A product required by eq. (4) is then given by:

$$\underline{D}\times\underline{A}=\begin{bmatrix}10&0&0&0&0\\0&10&0&0&0\\0&0&10&0&0\\0&0&0&10&0\\0&0&0&0&10\end{bmatrix}\begin{bmatrix}1&0&0&-1\\1&-1&0&0\\0&1&-1&0\\0&0&-1&1\\1&0&-1&0\end{bmatrix}=\begin{bmatrix}10&0&0&-10\\10&-10&0&0\\0&10&-10&0\\0&0&-10&10\\10&0&-10&0\end{bmatrix}$$

So based on eq. (4) and the solution vector, we can see that these elements will occupy the upper right hand corner of $\underline{A}_{eq}$. So that will take care of the last 4 columns in the first 5 rows.

But what about the first 8 columns? These are the elements in the line flow equations that multiply the variables P_{g1}, P_{g2}, P_{g4}, P_{B1}, P_{B2}, P_{B3}, P_{B4}, P_{B5}. Since we do not use the generation variables within the line flow equations, the first 3 columns of these top 5 rows will be zeros. The next 5 columns in these top 5 rows (columns 4-8) will also be zeros, except the one element in each of these rows that multiplies the corresponding line flow variable, and that element will be -1.

Finally, with respect to these top 5 equations, eq. (4) indicates that the right-hand-side will be 0 for each of them.

Thus, we can now write down all elements in the first 5 rows of our matrix, as follows:

$$\underline{A}_{eq}\underline{x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ - \end{bmatrix}$$

Now we need to write the last 4 equations. These are the DC power flow equations corresponding to eq. (5).

Again, we must remember that the solution vector contains all 4 angles, and therefore the DC power flow matrix needs to be a 4×4.

This augmented DC power flow matrix is given below:

$$\underline{B}' = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 20 & -10 & 0 \\ -10 & -10 & 30 & -10 \\ -10 & 0 & -10 & 20 \end{bmatrix}$$

So based on eq. (5) and the solution vector, we can see that this matrix will occupy the lower right hand side of the $\underline{A}_{eq}$ matrix. So that will take care of the last 4 columns in the bottom 4 rows. The resulting matrix appears as:

$$\underline{A}_{eq}\underline{x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & 30 & -10 & -10 & -10 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & -10 & 20 & -10 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & -10 & -10 & 30 & -10 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \_ \\ \_ \\ \_ \\ \_ \end{bmatrix}$$

Once again, we need to consider the first eight columns. Columns 4-8 correspond to the line flow variables, which do not appear in the DC power flow equations, so these will be zero.

$$\underline{A}_{eq}\,\underline{x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\ \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\ \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \_ \\ \_ \\ \_ \\ \_ \end{bmatrix}$$

The first three columns multiply the generation variables P_{g1}, P_{g2}, and P_{g4}. However, the DC power flow equations, eq. (5), require the negative of the *injections for all buses*, and the injections are the generation minus the load, i.e., P_{gk}-P_{dk}.

We do not have load variables P_{dk} included in the solution vector. In addition, we do not have generation for bus 3 (it is just a load bus) included in the solution vector. So what do we do?

The answer to this lies in recognizing that the "variables" we do not have in the solution vector, P_{d1}, P_{d2}, P_{d3}, P_{d4}, and P_{g3}, are not (when the electricity market does not allow demand bids) "variables" at all! In fact, they are known quantities, constants, given by:

$$P_{d1}=0,\ P_{d2}=1.0,\ P_{d3}=1.1787,\ P_{d4}=0,\ P_{g3}=0$$

Since these are constants, they can go to the right-hand side.
But with what sign? Eq. (5)

$$-\underline{P} + \underline{B}'\underline{\theta} = \underline{0} \qquad (5)$$

indicates that the injection, if modeled on the left-hand-side, should be negative, i.e., on the left-hand side, P_{gk}-P_{dk} should be negative. So we should see on the left-hand-side $-P_{gk}+P_{dk}$. But now we will take the load term onto the right-hand-side by subtracting it from both sides.

Thus, we see that the load term should show up on the right-hand-side as a negative number. This same logic also shows us that the elements multiplying the generation terms in the $\underline{A}_{eq}$ matrix should be -1.

So we are now prepared to complete the matrix relation for the equality constraints.

$$\underline{A}_{eq}\underline{x} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1.1787 \\ 0 \end{bmatrix}$$

Inequality constraints:

The inequality constraints are simple, as given in what follows:

$$
\begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ -500 \\ -500 \\ -500 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \leq \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}
$$

Solution by Matlab: The code for solving this linear program using Matlab is given below:

```
%Load is system load plus losses
Load=2.1787;

%Build objective function vector.
c=[1307 1211 1254 0 0 0 0 0 0 0 0 0]';

%Build Aeq matrix for equality constraints.
Aeq=[0  0  0  -1 0 0  0  0   10   0   0 -10;
    0  0  0  0 -1 0  0  0   10 -10   0   0;
    0  0  0  0 0 -1  0  0    0  10  -10  0;
    0  0  0  0 0 0  -1  0    0   0  -10 10;
    0  0  0  0 0 0   0 -1   10   0  -10  0;
   -1  0  0  0 0 0   0  0   30 -10  -10 -10
    0 -1  0  0 0 0   0  0  -10  20  -10  0;
    0  0  0  0 0 0   0  0  -10 -10   30 -10;
    0  0 -1  0 0 0   0  0  -10   0  -10 20;];

%Build right-hand side of equality constraint.
beq=zeros(9,1);
beq(7)=-1;
beq(8)=-1.1787;

%Build upper and lower bounds on decision variables.
LB=[.50   .375  .45     -500 -500 -500 -500 -500  -pi -pi -pi -pi]';
UB=[2.00  1.50  1.80     500  500  500  500  500   pi  pi  pi  pi]';
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=LINPROG(c,A,b,Aeq,beq,LB,UB);
```

The solution vector x is given by:

$$
\underline{x} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.2287 \\ 0.45 \\ -0.0152 \\ 0.0955 \\ 0.3242 \\ 0.4348 \\ 0.4197 \\ 0.0125 \\ 0.003 \\ -0.0295 \\ 0.0140 \end{bmatrix}, \quad Z=2705.8
$$

The solution is provided on the one-line diagram of Fig. 4.

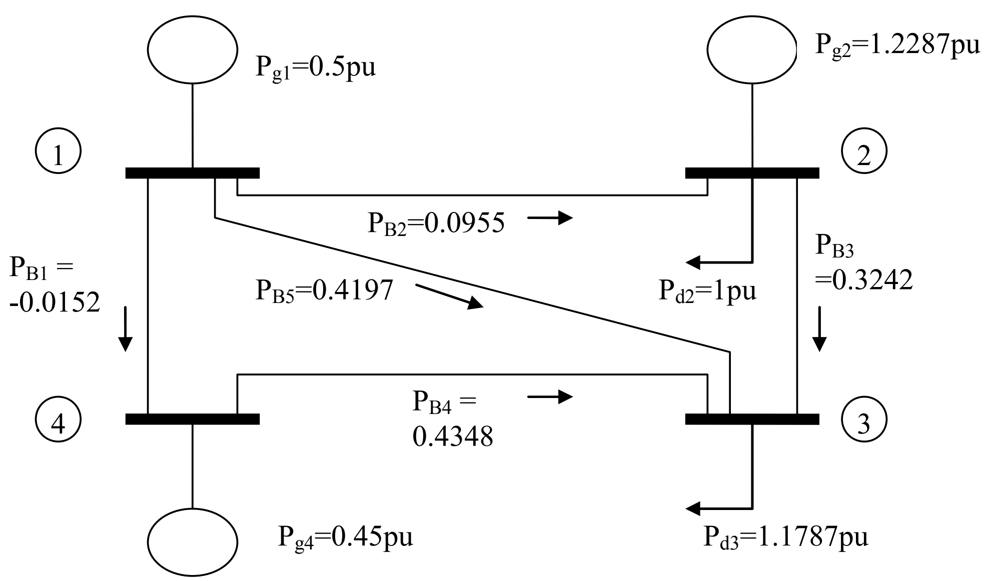

Fig. 4: Result in terms of generation levels and flows

One can easily check to see that the power is conserved at the buses.

The objective function that Matlab provides, is given above as Z=2705.8 $/hr.

It is of interest to compare this solution with a solution obtained from an economic dispatch (implying no representation of transmission). This problem will be

$$\min\ 1307P_{g1} + 1211P_{g2} + 1254P_{g3}$$

Subject to:

$$P_{g1} + P_{g2} + P_{g4} = 2.1787$$

$$0.5 \le P_{g1} \le 2.0$$

$$0.375 \le P_{g2} \le 1.5$$

$$0.45 \le P_{g4} \le 1.8$$

This is a LP that we can use Matlab or CPLEX to solve; however, it is actually a rather trivial solution. You can see that P_{g1} and P_{g4} are the more expensive units, and so we will take as little of those units as possible. Therefore we will assume

P_{g1}=0.5
P_{g4}=0.45

This means that P_{g2}= 2.1787-0.95=1.2287. The fact that this value of P_{g2} is feasible (between its limits) means that this must be the answer to the problem, since this is the maximum amount of the cheapest unit that we can have.

Compare this to the solution we obtained in our LPOPF, and you see it is the same!!!

Should we expect the solutions to be the same, given the network representation in the LOPF approach? The answer is NO, if the flows are at the branch capacities.

But if the flows are all within the branch capacities (and we are not representing losses), then the transmission system has no impact on the dispatch, and in this case, the solution will be identical to the solution that we obtained in the economic dispatch scenario.

The LPOPF solution is for all flows are within their branch capacities, since we have modeled all branch capacities to be 500 pu. In per-unit, this is effectively infinite branch capacity.

BASE CASE:
Now we will investigate the Lagrange multipliers for this loading level, assuming infinite capacity lines. These Lagrange multipliers, which are the same as the dual variables, are given in Table 2.

Table 2: Lagrange multipliers for P_{d2}=1.0, P_{d3}=1.1787 and infinite transmission capacity ($/per unit-hr)

Equality constraints		Lower bounds		Upper bounds	
Equation	Value*10^3	Variable	value	variable	value
P_{B1}	-0.0000	P_{g1}	96.0000	P_{g1}	0.0000
P_{B2}	-0.0000	P_{g2}	0	P_{g2}	0.0000
P_{B3}	0.0000	P_{g4}	43.0000	P_{g4}	0.0000
P_{B4}	-0.0000	P_{B1}	0	P_{B1}	0.0000
P_{B5}	-0.0000	P_{B2}	0	P_{B2}	0.0000
P_1	1.2110	P_{B3}	0	P_{B3}	0.0000
P_2	1.2110	P_{B4}	0	P_{B4}	0.0000
P_3	1.2110	P_{B5}	0	P_{B5}	0.0000
P_4	1.2110	θ_1	0	θ_1	0.0000
		θ_2	0	θ_2	0.0000
		θ_3	0	θ_3	0.0000
		θ_4	0	θ_4	0.0000

Lagrange multipliers on the last four equality constraints are very interesting to us, since they give the improvement in the objective function if we increase the right-hand-side of the corresponding equation by 1 unit. These are the so-called nodal prices, given in \$/per unit-hr. We see that the numbers are all \$1211/per unit-hr, and if we divide this by the power base of 100 MVA, we get \$12.11/MW-hr.

We also see that all Lagrange multipliers on the lower bounds are all zero, with the exception of the ones on P_{g1} and P_{g4}, which are 96 and 43, respectively, with units of \$/per unit-hr. Converting to \$/MW-hr, these values are 0.96 and 0.43, respectively, indicating the amount of improvement we can expect if we increase the corresponding right-hand-side of these inequalities by 1 unit. Since these equations look like, for example, $-P_{g1} \leq -0.5$, an increase in the right-hand-side corresponds to a decrease in the lower limit. Thus, for P_{g1}, if we move the lower limit from 50 MW to 49 MW, we can expect to improve the objective function by 96 cents per hour.

Lagrange multipliers for all other lower bounds, and for all upper bounds, are zero, since none of the other inequality constraints are binding.

CASE 1: P_{d2}=1.01, P_{d3}=1.1787, and infinite transmission capacity

We now make a slight modification, by changing the loading of bus 2 from 1.0 to 1.01, an increase of .01 per unit or 1 MW. The changes necessary to the Matlab code are:

```
%Build right-hand side of equality constraint. It will be vector of zeros
%except for element in first row, which is load-sum of minimum generation
beq=zeros(9,1);
beq(7)=-1.01;
beq(8)=-1.1787;
```

The solution is almost the same as in the base case, except for the objective function now evaluates to *Z*=2717.9. The previous objective function was Z=2705.8, an increase of 12.1 $/hr, confirming our understanding of the meaning of the nodal prices.

We find that a 1 MW increase in the loading anywhere in this network has the same effect on the objective function, to increase it by 12.1$/hr. Thus, nodal prices have no locational variation in this network. Any network with infinite transmission capacity will have this attribute (assuming no losses), since the transmission system effectively makes the entire system look like a single bus.

Case 2: P_{d2}=1.0, P_{d3}=1.1787, and 0.3 capacity constraint on P_{B3}
In this case, we maintain loading at the same levels as the base case, where we saw the flows as in Fig. 5:

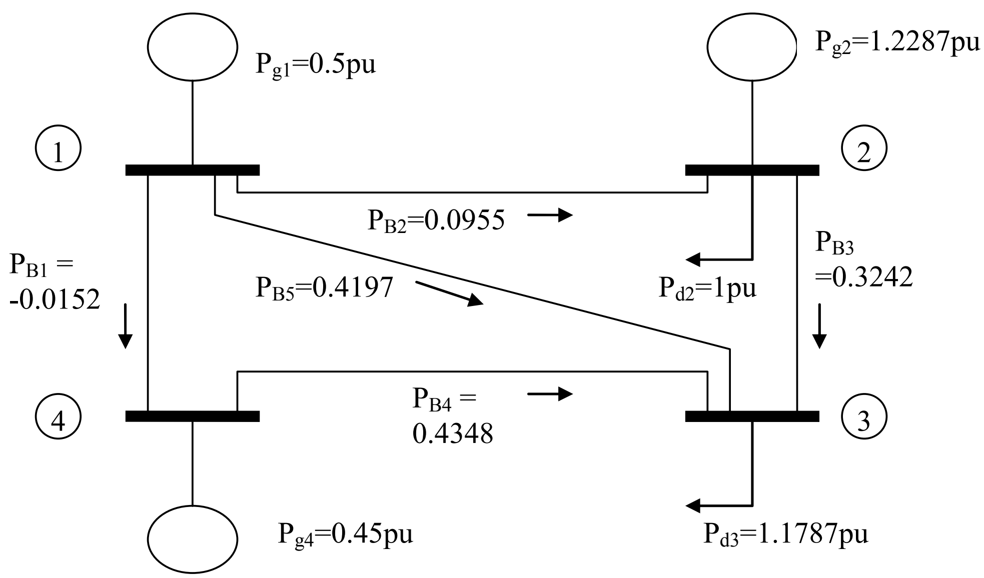

Fig. 5: Base case flows

We observe the flow on branch 3, P_{B3}=0.342. So let's consider that this branch has capacity of 0.3. This means that upper and lower bounds on P_{B3} should be changed from -500,500 to -0.3, 0.3. The resulting solution is given in Fig. 6:

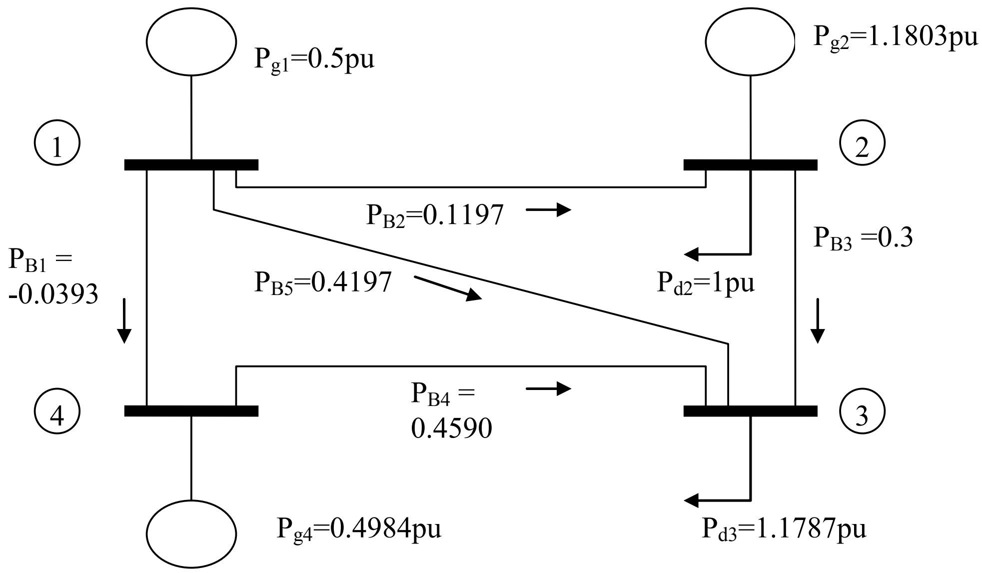

Fig. 6: Cases 1 flows

In comparing the previous two diagrams, we observe that

- The flow on branch 3 is constrained to 0.3 as desired.
- The flows all over the network have changed.
- The generation levels at buses 2 and 4 have changed.

Thus, the activation of a transmission constraint has changed the dispatch. This will affect the energy prices! This conclusion can be verified by looking at the Lagrange multipliers, given in the table below.

Table 3: Lagrange multipliers for P_{d2}=1.0, P_{d3}=1.1787 and infinite transmission capacity except for and 0.3 capacity constraint on P_{B3} (\$/per unit-hr)

Equality constraints		Lower bounds		Upper bounds	
Equation	Value*10^3	Variable	value	variable	value
P_{B1}	-0.0000	P_{g1}	63.7500	P_{g1}	0.0000
P_{B2}	-0.0000	P_{g2}	0.0000	P_{g2}	0.0000
P_{B3}	0.0860	P_{g4}	0.0000	P_{g4}	0.0000
P_{B4}	0.0000	P_{B1}	0.0000	P_{B1}	0.0000
P_{B5}	-0.0000	P_{B2}	0.0000	P_{B2}	0.0000
P_1	1.2432	P_{B3}	0.0000	P_{B3}	86.0000
P_2	1.2110	P_{B4}	0.0000	P_{B4}	0.0000
P_3	1.2647	P_{B5}	0.0000	P_{B5}	0.0000
P_4	1.2540	θ_1	0.0000	θ_1	0.0000
		θ_2	0.0000	θ_2	0.0000
		θ_3	0.0000	θ_3	0.0000
		θ_4	0.0000	θ_4	0.0000

Some comments about the Lagrange multipliers in Table 3:

1. Generation limits: We still see a non-zero Lagrange multiplier on the P_{g1} lower limit, as before, but the Lagrange multiplier on the P_{g4} lower limit has become zero, reflecting that P_{g4} had to increase and come off of its lower limit to compensate for the decrease in P_{g2} necessary to redispatch around the P_{B3} constraint.
2. Branch limits: The Lagrange multiplier on the P_{B3} upper bound is 86, and after dividing by 100 to change from per-unit to MW, it is 0.86 \$/MW-hr, reflecting the improvement in objective function that can be obtained from increasing the P_{B3} branch limit by 1 MW (from 0.30 per-unit to 0.31 per-unit).
3. Branch flows: The Lagrange multiplier on the P_{B3} flow (the P_{B3} equality constraint) is $0.086*10^3$=86, and after dividing by 100 to change from per-unit to MW, it is 0.86 \$/MW-hr, reflecting the improvement in objective function that can be obtained from

increasing the flow in this branch by 1 MW (from 0.30 per-unit to 0.31 per-unit). It makes sense that the Lagrange multiplier on the flow would be the same as the Lagrange multiplier on the branch limit.

4. Nodal prices: The Lagrange multipliers on the equality constraints corresponding to the 4 nodes are the nodal prices. Without transmission constraints, these prices were all the same, at 12.11 $/MW-hr, a price set entirely by the generator at bus 2 since it was the bus 2 generator that responded to any load change. But now they are all different, with only bus 2 price at 12.11 $/MW-hr. This difference reflects that, because of the transmission constraint, a load increase at one bus will incur a different cost than a load increase at another bus.

Comment #4 is worth investigating further. Let's increase the load at the highest price bus, bus #3, from 1.1787 to 1.1887 per unit, an increase of 1 MW. The resulting dispatch and flows are shown in Fig. 7. In order to gain intuition into what has happened, we have repeated Fig. 6 just below it so as to provide a convenient comparison.

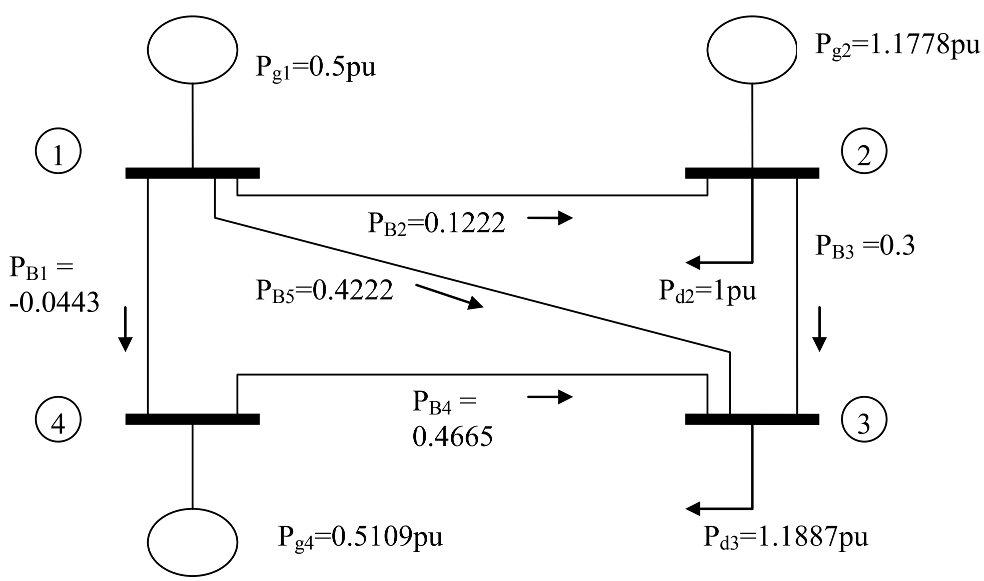

Fig. 7: Flows for case 2 with 1 MW increase in P_{d3}

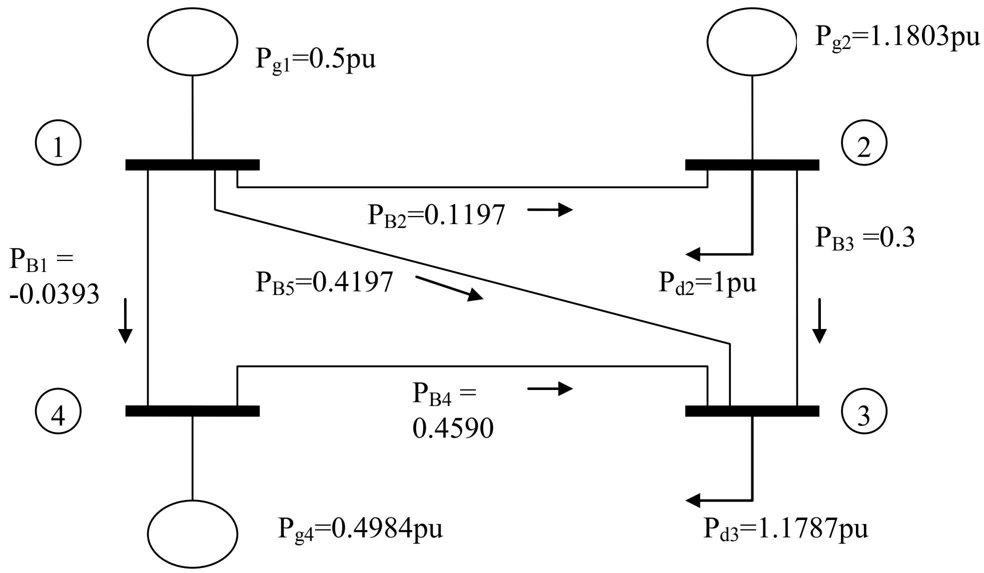

Fig. 6: Case 1 flows

The comparison shows that in order to supply an additional MW at bus 3, *the generation levels of 2 different units had to be modified.* Specifically, Unit 2 was decreased from 1.1803 to 1.1778, a decrease of 0.0025 per unit (0.25 MW) and Unit 4 was increased from 0.4984 per unit to 0.5109 per unit, an increase of 0.0125 (1.25 MW). Thus, Unit 4 was increased enough to supply the increased load at bus 3 and the decreased generation at bus 2.

Question: Why did we not just increase Unit 2 or increase Unit 4 by 1 MW?

Answer: Because the resulting flow on branch 3 would exceed its capacity!!!

In fact, it is not possible to supply additional load at bus 3 with only a single unit increase. We will always have to compensate for the load AND redispatch to compensate for the additional flow on the branch 3. As a result, the nodal price at bus 3 is a function of the generation costs at those buses that are used in the particular redispatch that achieves the minimum cost.

An Exercise:

1. Find the dispatch, flows, total cost, and all Lagrange multipliers for each of the following data (for each set of data, unless a data item is explicitly specified below, it is the same as in the "base case" that we identified above). Explain any surprising results.
   a. $P_{B1,max}$=0.01
   b. $P_{B2,max}$=0.08
   c. $P_{B4,max}$=0.40
   d. $P_{B5,max}$=0.40

Additional exercise (not required)

2. Set up a linear program in Matlab as done above to solve the linearized optimal power flow problem for the system and data given in Fig. 6.12 and Table 6.4, respectively, of the Kirchen text. Provide your Matlab code, the dispatch, flows, total cost, and all Lagrange multipliers.