Including the Consumer Function

What we did in the previous notes was solve the cost minimization problem. In these notes, we want to (a) see what such a solution means in the context of solving a social surplus maximization problem and (b) extend the formulation to include demand responsiveness.

1.0 Constant demand as consumer utility function

In the formulation of our cost minimization problem, we minimized the cost of supply subject to the requirement that the total supply equaled the fixed system demand. Here, our only decision variables were generation levels, i.e., demands were not decision variables but rather constants. This implies that demand is insensitive to price, i.e., demand is inelastic. This means that the consumer is not solving the following problem:

$$Max\ U(x)-px \qquad (1)$$

where U is the consumer's utility function (benefit from consuming x), dependent on x which is demand and p which is price, but rather is solving the following problem:

$$Max\ U(x) \qquad (2)$$

That is, the consumer is maximizing utility, which is a function of demand, but the consumer is totally ignoring the price when doing so. This is the way most residential consumers use electricity.

The consumer therefore determines how much demand they will consume via a local decision-making problem (e.g., "What do I need to do today"), and the social optimum can be solved treating the consumer demand as a constant. This means that the consumer demand function is a vertical line, as shown in Fig. 1.

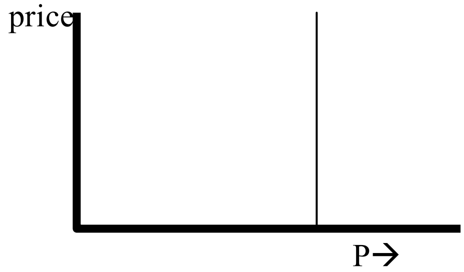


Fig. 1: Demand function for constant demand

Since the consumer demand is a constant, the consumer utility is also a constant. As a result, the social optimum is obtained via solution of a maximization problem that only includes generator cost functions (multiplied by -1) in the objective function. This is:

$$\text{Max: } \sum_k (-C_k(q_k)) \qquad (3)$$

Alternatively, the social optimum is obtained via solution of a minimization problem that only includes generator cost functions in the objective function. This is what we did in our LPOPF, and it is what utility companies have done for years.

$$\text{Min: } \sum_k C_k(q_k) \qquad (4)$$

Now, however, we want to account for the possibility that the consumer will watch their price and adjust their demand as a function of that price. In this case, we must include the consumers' utilities in the objective function, leading to the objective function used in the formulation for our present problem, which is:

$$\text{Max: } \sum_k (U_k(x_k) - C_k(q_k)) \qquad (5)$$

In both cases, we are maximizing the social surplus, the difference between the consumers' utilities and the suppliers' costs.

2.0 LPOPF with Consumer Utility

As in previous notes on LPOPF, we bring in network constraints, but this time we will do so with an objective function that maximizes social surplus. In addition, as with our LPOPF, we will use a piecewise linear approximation of the cost curves with only 1 "piece" per curve. Thus, each generation unit and each consumer is represented in the objective function by a constant times the MW output for that unit or the MW consumption by that consumer. So here is the formal statement of our problem:

$$\textbf{min} \quad \sum_{k \in \{generator\_buses\}} s_{gk} P_{gk} + \sum_{k \in \{load\_buses\}} - s_{dk} P_{dk} \qquad (6)^{1}$$

Subject to:

$$\underline{P} = \underline{B}' \underline{\theta} \qquad (7)^{2}$$

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \qquad (8)^{3}$$

$$-\underline{P}_{B,\max} \leq \underline{P}_B \leq \underline{P}_{B,\max} \qquad (9)^{4}$$

$$0 \leq P_{gk} \leq P_{gk,\max}, \forall k \in \{generator\_buses\} \qquad (10)^{5}$$

$$\mathbf{0 \leq P_{dk} \leq P_{dk,\max}, \forall k \in \{load\_buses\}} \qquad (11)^{6}$$

where

$$P_k = P_{gk} - P_{dk}, k = 1,...N \qquad (12)^{7}$$

---

[1] We want to maximize social surplus as defined by $\sum U_k - \sum C_k$, but this is the same as minimizing $\sum C_k - \sum U_k$. We make this change because the LP available to us in Matlab is a minimizing LP.

[2] So these are the DC power flow equations to represent the network. However, we must include all nodal injections $P_1,\ldots P_N$ and all angles $\theta_1 \ldots \theta_N$ in this set of equations.

[3] These are the equation to get the line flows. Again, we need to include all angles $\theta_1 \ldots \theta_N$ in this set of equations.

[4] These are the limits on the line flows. Notice that there is only one circuit rating, but it must be enforced as a limit if the flow is in one direction or in the other.

[5] These are the limits on the linear cost curve variables.

[6] The limits on the linear demand curve variables.

[7] This equation relates the variables used in the cost curves (P_{kj}) to the variables used in the DC power flow equations (P_k).

We identify the decision vector as:

$$\underline{x} = \begin{bmatrix} P_{g1} \\ \vdots \\ P_{gn} \\ P_{d1} \\ \vdots \\ P_{dn} \\ P_{b1} \\ \vdots \\ P_{bm} \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}; \quad \underline{c} = \begin{bmatrix} s_{g1} \\ \vdots \\ s_{gn} \\ -s_{d1} \\ \vdots \\ -s_{dn} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{13}$$

We are now in a position to state the LPOPF more compactly.

$$\text{Max } \underline{c}^T \underline{x} \text{ Subject to:} \tag{14}$$

$$\underline{A}_{eq}\underline{x} = \underline{b}_{eq}, \qquad \underline{x}_{\min} \le \underline{x} \le \underline{x}_{\max} \tag{15}$$

where the equality constraints in the A_{eq} matrix equation model the line flow equations and DC power flow equations.

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \tag{16}$$

$$-\underline{P} + \underline{B}'\underline{\theta} = \underline{0} \tag{17}$$

and the inequality constraints are given by:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ -P_{b1,ma} \\ \vdots \\ -P_{bm,\max} \\ -\pi \\ \vdots \\ -\pi \end{bmatrix} \leq \begin{bmatrix} P_{g1} \\ \vdots \\ P_{gn} \\ P_{d1} \\ \vdots \\ P_{dn} \\ P_{b1} \\ \vdots \\ P_{bm} \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} \leq \begin{bmatrix} P_{g1\max} \\ \vdots \\ P_{gn,\max} \\ P_{d1,\max} \\ \vdots \\ P_{dn,\max} \\ P_{b1,\max} \\ \vdots \\ P_{bm,\max} \\ \pi \\ \vdots \\ \pi \end{bmatrix} \tag{18}$$

Some particular notes about the above problem statement:

- The upper right-hand m×n submatrix of $\underline{A}_{eq}$ is $\underline{D}\times\underline{A}$.
- The lower right-hand n×n submatrix of $\underline{A}_{eq}$ is $\underline{B}'$.

- The right-hand-side of the equality constraint equation, $\underline{b}_{eq}$, is all zeroes because we now have variables for the demand which means it must be included in the $\underline{A}_{eq}$ matrix instead of being a fixed constant (and therefore represented in the $\underline{b}_{eq}$ vector).

3.0 Example: Unconstrained transmission

We illustrate using an example that is similar to the example used in the LPOPF notes, which is an extension of the previous example. The one-line diagram for the example system is given in Fig. 2.

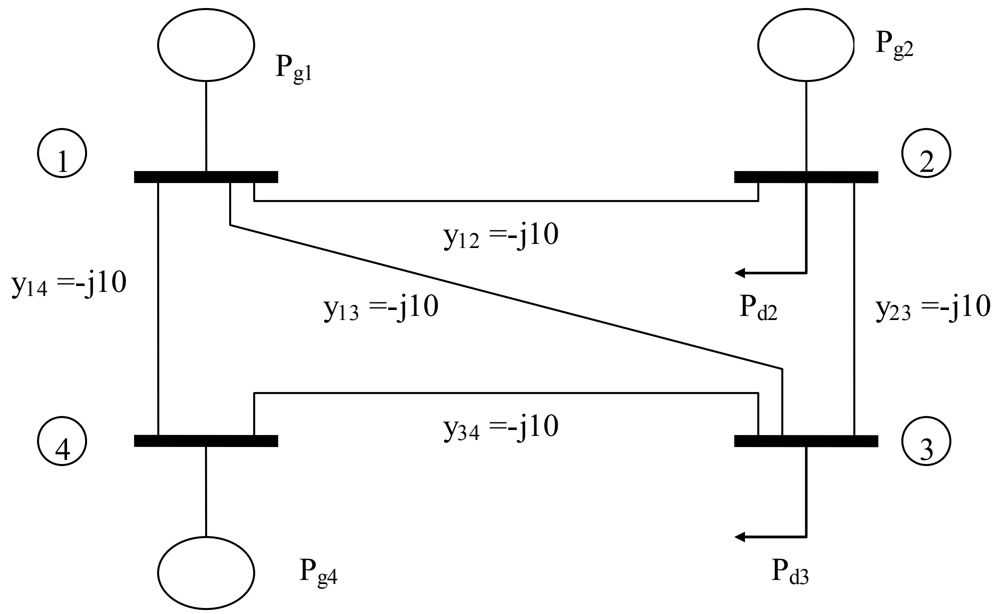


Fig. 2: One line diagram for example system

We will use the same data for the unit cost-curves as we did in the LPOPF notes. These were

$$K_1(P_{g1}) = s_{g1}P_{g1}$$

$$K_2(P_{g2}) = s_{g2}P_{g2}$$

$$K_4(P_{g4}) = s_{g4}P_{g4}$$

where the generation variables are in pu and the coefficients are

s_{g1}=1307 \$/pu-hr
s_{g2}=1211 \$/pu-hr
s_{g4}=1254 \$/pu-hr

The constraints are

$50 \le P_{g1} \le 200$

$37.5 \le P_{g2} \le 150$

$45 \le P_{g4} \le 180$

We use the following linearized functions for demand bids:

$$D_2(P_{d2}) = s_{d2} P_{d2}$$

$$D_3(P_{d3}) = s_{d3} P_{d3}$$

where the load variables are in per-unit and the coefficients are

s_{d2}=-1300 \$/pu-hr

s_{d3}=-1200 \$/pu-hr

The constraints are

$100 \le P_{d2} \le 200$

$200 \le P_{d3} \le 300$

<u>Objective function</u>: Let's explicitly write out the solution vector.

$$\underline{x} = \begin{bmatrix} \underline{P}_g \\ \underline{P}_d \\ \underline{P}_B \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

So, using these coefficients, the objective function is:

$$\underline{Z(\underline{x}) = c^T} \, \underline{x} = \begin{bmatrix} 1307 & 1211 & 1254 & -1300 & -1200 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

<u>Equality constraints</u>: The equality constraints are given in eqs. (16) and (17), repeated here for convenience:

$$-\underline{P}_B + (\underline{D} \times \underline{A}) \times \underline{\theta} = \underline{0} \qquad (16)$$

$$-\underline{P} + \underline{B}'\underline{\theta} = \underline{0} \qquad (17)$$

We need to build all of these equality constraints into a matrix form of $\underline{A}_{eq}\underline{x}=\underline{b}_{eq}$. We begin by noting dimensions.

- Columns: Since the solution vector $\underline{x}$ is 14x1, $\underline{A}_{eq}$ must have 14 columns in order to pre-multiply $\underline{x}$.

- Rows: Since there are 5 branches, eq. (16) will contribute 5 rows to $\underline{A}_{eq}$. Since there are 4 buses, eq. (17) will contribute 4 rows to $\underline{A}_{eq}$. So $\underline{A}_{eq}$ will have total of 9 rows.

Therefore, the dimensions of $\underline{A}_{eq}$ will be 9×14.

We begin with the line flow equations, eq. (16). From the notes on "Power flow equations," we can recall the $\underline{D}$ and $\underline{A}$ matrix. The $\underline{D}$ matrix is exactly the same as before, which is:

$$\underline{D} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

And the node-arc incidence matrix, $\underline{A}$, is:

$$\underline{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$$

The $\underline{D}$x$\underline{A}$ produce required by eq. (4) is then given by:

$$\underline{D} \times \underline{A} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & -10 \\ 10 & -10 & 0 & 0 \\ 0 & 10 & -10 & 0 \\ 0 & 0 & -10 & 10 \\ 10 & 0 & -10 & 0 \end{bmatrix}$$

So based on eq. (4) and the solution vector, we can see that these elements will occupy the upper right hand corner of $\underline{A}_{eq}$. So that will take care of the last 4 columns in the first 5 rows.

But what about the first 10 columns? These are the elements in the line flow equations that multiply the variables P_{g1}, P_{g2}, P_{g4}, P_{d2}, P_{d3}, P_{B1}, P_{B2}, P_{B3}, P_{B4}, P_{B5}.

Since we do not use the generation or demand variables within the line flow equations, the first 5 columns of these top 5 rows will be zeros. The last 5 columns in these top 5 rows will also be zeros, except the one element in each of these rows that multiply the corresponding line flow variable, and that element will be -1.

Finally, with respect to these top 5 equations, eq. (16) indicates that the right-hand-side will be 0 for each of them.

Thus, we can now write down all elements in the first 5 rows of our matrix, as follows:

$$
\underline{A}_{eq}\,\underline{x} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & -
\end{bmatrix}
\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Now we need to write the last 4 equations. These are the DC power flow equations corresponding to eq. (17).

Again, we must remember that the solution vector contains all 4 angles, and therefore the DC power flow matrix needs to be a 4x4.

This augmented DC power flow matrix is given below:

$$\underline{B}' = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 20 & -10 & 0 \\ -10 & -10 & 30 & -10 \\ -10 & 0 & -10 & 20 \end{bmatrix}$$

So based on eq. (17) and the solution vector, we can see that this matrix will occupy the lower right hand side of the $\underline{A}_{eq}$ matrix. So that will take care of the last 4 columns in the bottom 4 rows. The resulting matrix appears as:

$$\underline{A}_{eq}\,\underline{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & 30 & -10 & -10 & -10 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & -10 & 20 & -10 & 0 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & -10 & -10 & 30 & -10 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Once again, we need to consider the first eight columns. Columns 6-10 correspond to the line flow variables, which do not appear in the DC power flow equations, so these will be zero.

$$\underline{A}_{eq}\underline{x}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ \_ & \_ & \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ \_ & \_ & \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\ \_ & \_ & \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\ \_ & \_ & \_ & \_ & \_ & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 \end{bmatrix}\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first three columns multiply the generation variables P_{g1}, P_{g2}, and P_{g4}, and columns 4 and 5 multiply the load variables P_{d2} and P_{d3}.

However, the DC power flow equations, eq. (17), require the negative of the *injections for all buses*, and the injections are the generation minus the load, i.e., P_{gk}-P_{dk}. So we want to model $-P_{gk}+P_{dk}$ on the left-hand-side in the last 4 rows. This will be done by placing a -1 and +1 in the appropriate place.

$$\underline{A}_{eq}\,\underline{x}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 10 & 0 & -10 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & -10 & -10 & -10 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -10 & -10 & 30 & -10 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -10 & 20 \end{bmatrix}\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Inequality constraints</u>: The inequality constraints are simple, as given below. Notice that the -500 to 500 constraints on line flows imply we are modeling no transmission constraints.

$$
\begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ 1 \\ 2 \\ -500 \\ -500 \\ -500 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \le \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \le \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 2 \\ 3 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}
$$

Solution by Matlab: The code for solving this linear program using Matlab is given below:

```
%Build objective function vector.
c=[1307 1211 1254 -1300 -1200 0 0 0 0 0 0 0 0 0]';

%Build A matrix for inequality constraints Ax<b.
A=[];
%Build b, the right-hand-side of inequality constraints.
b=[];

%Build Aeq matrix for equality constraints.
```

```
Aeq=[0  0  0  0  0 -1 0 0  0  0   10   0   0 -10;
     0  0  0  0  0  0 -1 0  0  0   10 -10   0   0;
     0  0  0  0  0  0 0 -1  0  0    0  10  -10  0;
     0  0  0  0  0  0 0 0  -1  0    0   0  -10 10;
     0  0  0  0  0  0 0 0   0 -1   10   0  -10  0;
    -1  0  0  0  0  0 0 0   0  0   30  -10  -10 -10
     0 -1  0  1  0  0 0 0   0  0  -10  20  -10  0;
     0  0  0  0  1  0 0 0   0  0  -10 -10   30 -10;
     0  0 -1  0  0  0 0 0   0  0  -10   0  -10 20;];

%Build right-hand side of equality constraint. It will be vector of
zeros
beq=zeros(9,1);

%Build upper and lower bounds on decision variables.
LB=[.50   .375  .45   1 2  -500 -500 -500 -500 -500  -pi -pi -pi -pi]';
UB=[2.00  1.50  1.80  2 3   500  500  500  500  500   pi  pi  pi  pi]';
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=LINPROG(c,A,b,Aeq,beq,LB,UB);
%'X=', X,FVAL,'eqaulity', LAMBDA.eqlin, 'upper', LAMBDA.upper, 'lower',
LAMBDA.lower
%
% Compute the dollars paid to each participant:
dollars=c.*X;
%Write out the dollars paid to each participant
dollars
```

The solution vector <u>x</u> is given below. The limits on the variables are also repeated here so that it is easy to see which ones are at their limit.

$$\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 1.8 \\ 1.8 \\ 2.0 \\ -0.5875 \\ 0.4625 \\ 0.1625 \\ 1.2125 \\ 0.6250 \\ 0.0125 \\ -0.0338 \\ -0.05 \\ 0.0712 \end{bmatrix} \qquad \begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ 1 \\ 2 \\ -500 \\ -500 \\ -500 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \le \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \le \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 2 \\ 3 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

The solution is provided in Fig. 3.

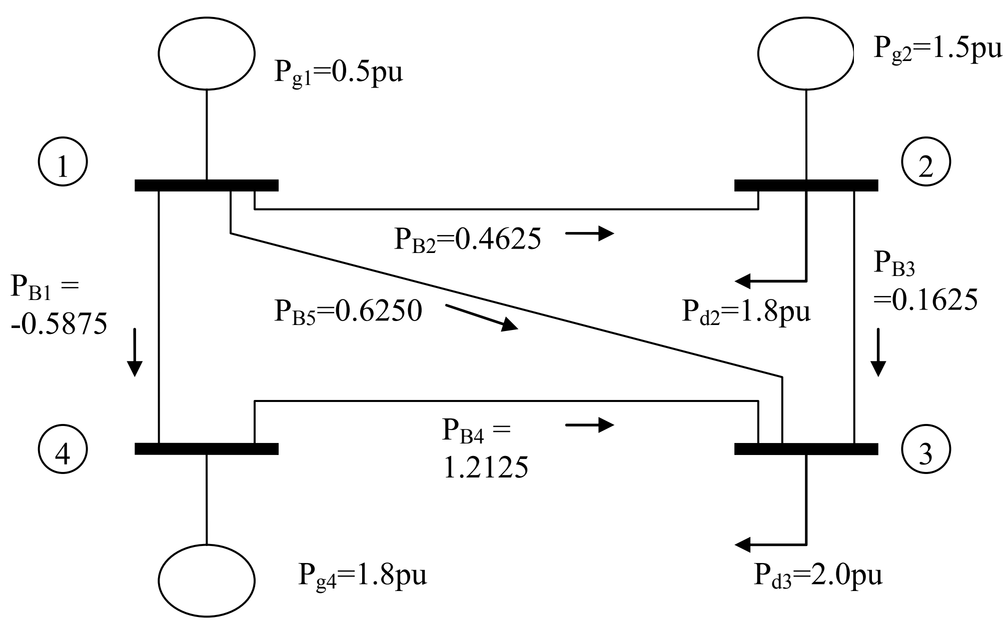


Fig. 3

One can easily check to see that the power is conserved at the buses.

Objective function value: The objective function that Matlab provides (FVAL) is Z=-12.8 \$/hr. This is negative of the social surplus (Matlab requires all problems to be minimization problems, so we had to minimize the negative of the social surplus in order to maximize social surplus).

So the social surplus (Total Utility of Load less Total Cost of Supply) is \$12.80. Not too much! This is because the demanders are valuing the energy at just a little above cost.

- If we changed the utility function coefficients to -1500 and -1400, from -1300 and -1200, respectively, the social surplus would change to \$904/hr.
- If we changed utility function coefficients to -1000 and -900, respectively, the social surplus would be -\$924/hr, indicating the cost of supply is more than the utility of

consumption, and the only reason any power is being consumed is the lower bound constraints we have placed on generation and demand.

<u>Lagrange multipliers</u>: Now let's investigate Lagrange multipliers for this case, assuming infinite capacity lines. These Lagrange multipliers (the same as the dual variables), are given in Table 2.

Table 2: Lagrange multipliers for infinite transmission capacity

Equality constraints		Lower bounds		Upper bounds	
Equation	Value*10^3	Variable	value	variable	value
P_{B1}	-0.0000	P_{g1}	7.0000	P_{g1}	0.0000
P_{B2}	-0.0000	P_{g2}	0.0000	P_{g2}	89.0000
P_{B3}	0.0000	P_{g4}	0.0000	P_{g4}	46.0000
P_{B4}	0.0000	P_{d2}	0.0000	P_{d2}	0.0000
P_{B5}	-0.0000	P_{d3}	100.0000	P_{d2}	0.0000
P_1	1.3000	P_{B1}	0.0000	P_{B1}	0.0000
P_2	1.3000	P_{B2}	0.0000	P_{B2}	0.0000
P_3	1.3000	P_{B3}	0.0000	P_{B3}	0.0000
P_4	1.3000	P_{B4}	0.0000	P_{B4}	0.0000
		P_{B5}	0.0000	P_{B5}	0.0000
		θ_1	0.0000	θ_1	0.0000
		θ_2	0.0000	θ_2	0.0000
		θ_3	0.0000	θ_3	0.0000
		θ_4	0.0000	θ_4	0.0000

Lagrange multipliers on the last 4 equality constraints are very interesting, since they give the improvement in the objective function if we increase the right-hand-side of the corresponding equation by 1 unit. These are the nodal prices, given in $/per unit-hr. The numbers are all $1300/per unit-hr; if we divide this by the power base (100 MVA), we get $13.00/MW-hr. This is also the coefficient of the demand at bus 2, P_{d2}.

Now let's consider the Lagrange multipliers:

- Lower bounds: P_{g1} and P_{d3} are non-zero, indicating they are at their lower bounds, as confirmed by decision vector on pg. 22.
- Upper bounds: P_{g2} and P_{g4} are non-zero, indicating they are at their upper bounds, as confirmed by decision vector on pg. 22.
- Not constrained (regulating): Only P_{d2} has 0 Lagrange multipliers for both lower and upper bounds, indicating it is not at either bound (this variable is "regulating").

Connection! There is only ONE unconstrained variable, P_{d2}, and it is also the variable that is setting the nodal prices ($13.00/MW-hr) throughout the network! A look at the coefficients will show why:

s_{g2}=1211 $/pu-hr s_{d2}=-1300 $/pu-hr
s_{g4}=1254 $/pu-hr s_{d3}=-1200 $/pu-hr
s_{g1}=1307 $/pu-hr

Think of the algorithm like this:

- It first sets generation and load at lower limits (there is no choice about this much supply and demand). One variable must come off its lower bound in order to provide power balance. Since sum of load lower bounds is 3, and sum of gen lower bounds is 1.37, one or more of the gens must come off their lower bounds by 1.63 in order to provide a feasible solution. This gen will be the least expensive one(s). In this case, it is G2 and G4. (G2 gets pushed to its limit in this step)
- Then it takes a MW of supply and a MW of demand from the gen/load pair that is not at

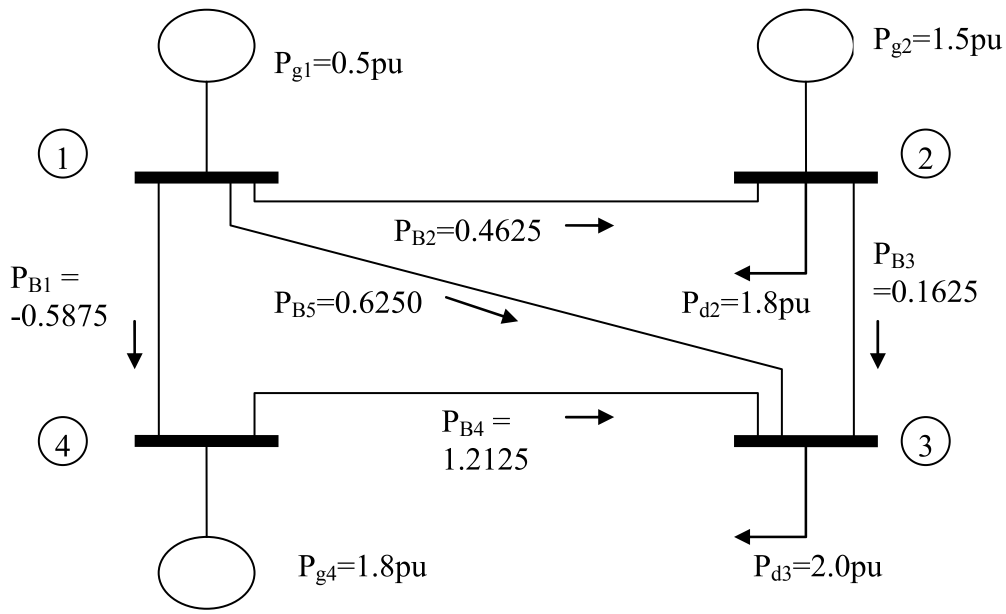
upper bounds and provides the most positive surplus. This will be the gen with the least cost and the load with the greatest utility, as long as the surplus is positive. In our example, the first gen/load pair taken, after finding a feasible solution, are G4/D2.

- As soon as either the gen or the load of the max-surplus gen/load pair reaches its upper limit, it will replace that gen or load with the one that yields the next largest surplus. In our case, G4 reaches its upper limit first, and it tries to replace it with G1. But the G1/D2 pair has coefficients that result in a negative surplus! So the maximum surplus is found when G4 reaches its upper limit.

You should be able to see that the algorithm will always terminate with just one gen or load regulating, and that gen or load will set the nodal price throughout the network (for the unconstrained transmission case).

4.0 Example: Constrained transmission

We will constrain the transmission on branch 3. Reference to the old solution of Fig. 3, repeated here for convenience, indicates that the flow on branch 3 is 0.1625. So we will constrain that flow to be 0.16.



The Matlab code for this is given below.

```
%Build objective function vector.
c=[1307 1211 1254 -1300 -1200 0 0 0 0 0 0 0 0 0]';

%Build A matrix for inequality constraints Ax<b.
A=[];
%Build b, the right-hand-side of inequality constraints.
b=[];

%Build Aeq matrix for equality constraints.
Aeq=[0  0  0  0  0 -1 0 0  0  0   10   0   0 -10;
    0  0  0  0  0  0 -1 0  0  0   10 -10   0   0;
    0  0  0  0  0  0 0 -1  0  0    0  10  -10  0;
    0  0  0  0  0  0 0 0  -1  0    0   0  -10 10;
```

```
    0  0  0  0  0  0 0 0   0 -1   10   0  -10  0;
   -1  0  0  0  0  0 0 0   0  0   30  -10  -10 -10
    0 -1  0  1  0  0 0 0   0  0  -10  20  -10  0;
    0  0  0  0  1  0 0 0   0  0  -10 -10   30 -10;
    0  0 -1  0  0  0 0 0   0  0  -10   0  -10 20;];

%Build right-hand side of equality constraint. It will be vector of zeros
beq=zeros(9,1);

%Build upper and lower bounds on decision variables.
LB=[.50   .375  .45   1 2  -500 -500 -0.16 -500 -500  -pi -pi -pi -pi]';
UB=[2.00  1.50  1.80  2 3   500  500  0.16  500  500   pi  pi  pi  pi]';
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=LINPROG(c,A,b,Aeq,beq,LB,UB);
%'X=', X,FVAL,'eqaulity', LAMBDA.eqlin, 'upper', LAMBDA.upper, 'lower', LAMBDA.lower
%
% Compute the dollars paid to each participant:
dollars=c.*X;
%Write out the dollars paid to each participant
dollars
```

The "new" and the "old" decision vectors are provided below, together with the limits.

New solution   Old solution   Limits

$$\begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0.5067 \\ 1.5 \\ 1.8 \\ 1.8067 \\ 2.0 \\ -0.5867 \\ 0.4667 \\ 0.1600 \\ 1.2133 \\ 0.6267 \\ 0.0127 \\ -0.0340 \\ -0.05 \\ 0.0713 \end{bmatrix} \quad \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 1.8 \\ 1.8 \\ 2.0 \\ -0.5875 \\ 0.4625 \\ 0.1625 \\ 1.2125 \\ 0.6250 \\ 0.0125 \\ -0.0338 \\ -0.05 \\ 0.0712 \end{bmatrix} \quad \begin{bmatrix} 0.5 \\ 0.375 \\ 0.45 \\ 1 \\ 2 \\ -500 \\ -500 \\ -0.16 \\ -500 \\ -500 \\ -\pi \\ -\pi \\ -\pi \\ -\pi \end{bmatrix} \le \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g4} \\ P_{d2} \\ P_{d3} \\ P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \le \begin{bmatrix} 2 \\ 1.5 \\ 1.8 \\ 2 \\ 3 \\ 500 \\ 500 \\ 0.16 \\ 500 \\ 500 \\ \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

The new solution is provided in Fig. 4.

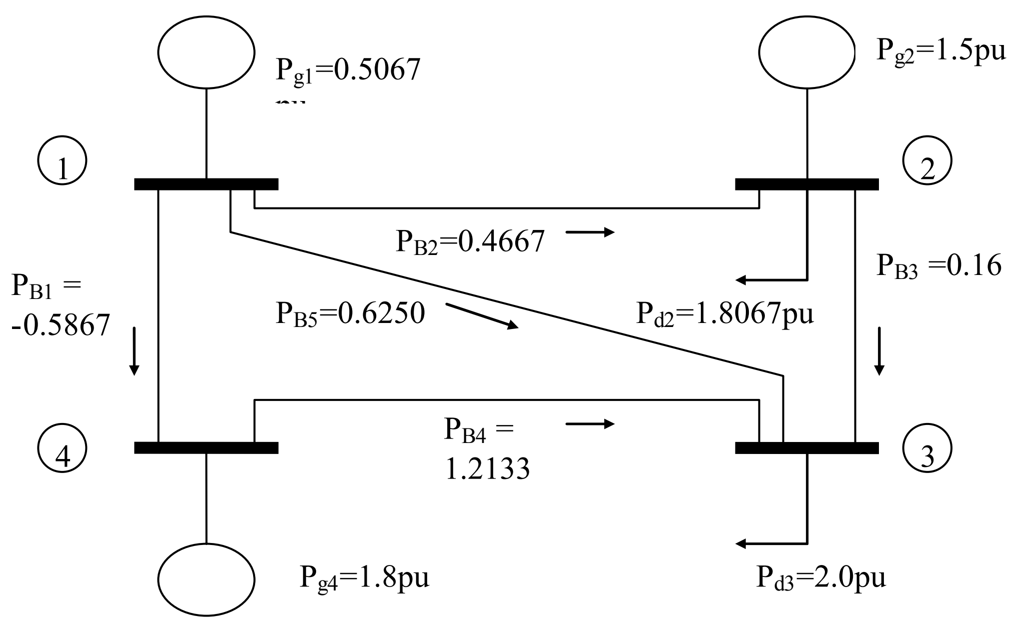


Fig. 4

Objective function value: The objective function that Matlab provided (FVAL) in the unconstrained case was Z=-12.8 $/hr (social surplus of $12.80/hr). Now in the constrained case it is Z=-$12.75/hr (social surplus of $12.75/hr).

The social surplus has decreased, confirming the principle that adding new constraints can never result in an improvement in the objective function.

Lagrange multipliers: The Lagrange multipliers (the same as the dual variables), are given in Table 4.

Table 4: Lagrange multipliers for constrained transmission capacity

Equality constraints		Lower bounds		Upper bounds	
Equation	Value*10^3	Variable	value	variable	value
P_{B1}	-0.0000	P_{g1}	0.0000	P_{g1}	0.0000
P_{B2}	0.0000	P_{g2}	0.0000	P_{g2}	89.0000
P_{B3}	0.0187	P_{g4}	0.0000	P_{g4}	55.3333
P_{B4}	0.0000	P_{d2}	0.0000	P_{d2}	0.0000
P_{B5}	0.0000	P_{d3}	111.6667	P_{d3}	0.0000
P_1	1.3070	P_{B1}	0.0000	P_{B1}	0.0000
P_2	1.3000	P_{B2}	0.0000	P_{B2}	0.0000
P_3	1.3117	P_{B3}	0.0000	P_{B3}	18.6667
P_4	1.3093	P_{B4}	0.0000	P_{B4}	0.0000
		P_{B5}	0.0000	P_{B5}	0.0000
		θ_1	0.0000	θ_1	0.0000
		θ_2	0.0000	θ_2	0.0000
		θ_3	0.0000	θ_3	0.0000
		θ_4	0.0000	θ_4	0.0000

Some observations:

1. In the unconstrained case, all four nodal prices were $13/MWhr, now, in the constrained case, only bus 2 is $13/MWhr (set by P_{d2}), which is a regulating (not at a limit) unit. And all of the remaining nodal prices are different.

2. P_{g1} is also regulating, and therefore the bus 1 nodal price is set by the P_{g1} bid which was $13.07/hr.
3. Buses 3 and 4 have load or generation at a limit. Bus 3 has P_{d3} at its lower limit, and bus 4 has P_{g4} at its upper limit. So neither of these buses are regulating. Notice that the nodal prices at these buses are different from the cost or utility function coefficient at the bus:
   - Bus 3 has utility function coefficient of 1200 whereas its LMP is 1311.70
   - Bus 4 has cost function coefficient of 1254 whereas its LMP is 1309.30.

   This shows that buses with regulating units or demands set their own price, whereas non-regulating buses have prices set by other buses in the network.
4. If there were no binding transmission constraints (effectively an infinite transmission capacity situation), then the prices at buses 3 and 4 would be set by one other bus in the network. But with a

binding transmission constraint (i.e., presence of congestion), then the prices will be set by the units needed to supply an additional MW at the bus AND maintain flow within the limit. As we have seen before, this will necessarily involve more than one unit.