Additional Sensitivity Information

1.0 Efficient computation of GSFs

In the previous discussion, it was assumed that we would be able to compute $(B')^{-1}$, i.e., that the number of nodes would not be too large, which is can be the case under some approximations such as those made by the IDC [**Error! Bookmark not defined.**]. However, it is also common that this is not the case, i.e., that we may want to obtain GSFs for a system where the number of nodes is very large.

In such a case, one can obtain the GSFs but only for one shift at a time, via

$$\Delta \underline{P} = \underline{B}' \Delta \underline{\theta} \tag{14}$$

$$\Delta \underline{P}_B = (\underline{D} \times \underline{A}) \times \Delta \underline{\theta} \tag{15}$$

Equation (14) is solved for $\Delta\underline{\theta}$ via LU factorization for a given $\Delta\underline{P}$, and then the resulting $\Delta\underline{\theta}$ is used in (15) to obtain the line flow shifts in $\Delta\underline{P}_B$.

Example 4: Repeat example 1, which is to obtain the GSF for all branches corresponding to an increase in bus 2 injection and a decrease in bus 3 injection.

$$\Delta \underline{P} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} = \underline{B}' \Delta \underline{\theta}$$

Using LU factorization:

$$\begin{bmatrix} 1 & -0.5 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix} \qquad L = \begin{bmatrix} 20 & 0 & 0 \\ -10 & & \\ 0 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 25 & -10 \\ 0 & -10 & 20 \end{bmatrix} \qquad L = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 25 & \\ 0 & -10 & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & -10 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 16 \end{bmatrix} \qquad L = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 25 & 0 \\ 0 & -10 & 16 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now use backwards/forwards substitution to obtain $\Delta\underline{\theta}$, resulting in

$$Lw = b$$

$$\Rightarrow \begin{bmatrix} 20 & 0 & 0 \\ -10 & 25 & 0 \\ 0 & -10 & 16 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$w_1 = 0.05$$

$$w_2 = (-1 + 10 * 0.05) / 25 = -0.02$$

$$w_3 = (10/16) * (-0.02) = -0.0125$$

$$U = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ux = w$$

$$\Rightarrow \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.05 \\ -0.02 \\ -0.0125 \end{bmatrix}$$

$$x_3 = -0.0125$$

$$x_2 = -0.02 + 0.4 * (-0.0125) = -0.0250$$

$$x_1 = 0.05 + 0.5 * (-0.0250) = 0.0375$$

And this gives our angle changes as

$$\begin{bmatrix} \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} = \begin{bmatrix} 0.05 \\ -0.02 \\ -0.0125 \end{bmatrix}$$

Now we can use eq. (15) to obtain

$$\Delta \underline{P}_B = (\underline{D} \times \underline{A}) \times \Delta\underline{\theta}$$

$$= \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.05 \\ -0.02 \\ -0.0125 \end{bmatrix} = \begin{bmatrix} 0.125 \\ -0.375 \\ 0.625 \\ 0.125 \\ 0.25 \end{bmatrix}$$

which is in agreement with the result of example 1.

2.0 Line outage distribution factors

The line outage distribution factors (LODFs) are linear estimates of the ratio:

change in flow on circuit ℓ due to
outage of circuit k, denoted by Δf_ℓ,
to
pre-contingency flow on
circuit k, denoted by f_{k0}.

In other words, it provides the fraction of pre-contingency flow on circuit k that appears on circuit ℓ following outage of circuit k, and is given by

$$d_{\ell,k} = \Delta f_\ell \; / \; f_{k0} \tag{16}$$

It is then clear that the change in flow on circuit ℓ due to the outage of circuit k is given by

$$\Delta f_\ell = d_{\ell,k} \times f_{k0} \tag{17}$$

The derivation is lengthy; we will not go through it here. To understand the result, we define a matrix X’ such that

$$\underline{X}' = (\underline{B}')^{-1} \tag{18}$$

This means that

$$\Delta \underline{P} = \underline{B}' \Delta \underline{\theta} \Rightarrow \Delta \underline{\theta} = \underline{X}' \Delta \underline{P} \tag{19}$$

Then we define another matrix X such that it is the same as X’ except we append another row at the top and another column to the left, corresponding to the reference bus (assumed bus #1) injection and angle, as shown below:

$$\underline{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \underline{X}' & \\ 0 & & & \end{bmatrix} \tag{20}$$

The line outage distribution factor $d_{\ell,k} = \Delta f_\ell \; / \; f_{k0}$ corresponding to the additional flow on branch k from outage of branch ℓ is then given by

$$d_{\ell,k} = \frac{\dfrac{x_k}{x_\ell}\left(X_{in} - X_{jn} - X_{im} + X_{jm}\right)}{x_k - \left(X_{nn} + X_{mm} - 2X_{nm}\right)} \tag{21}$$

In (21),

- x_k and x_ℓ are the reactances of outage branch k and remaining branch ℓ, respectively;
- m and n are bus numbers terminating branch k;
- i and j are bus numbers terminating branch ℓ.

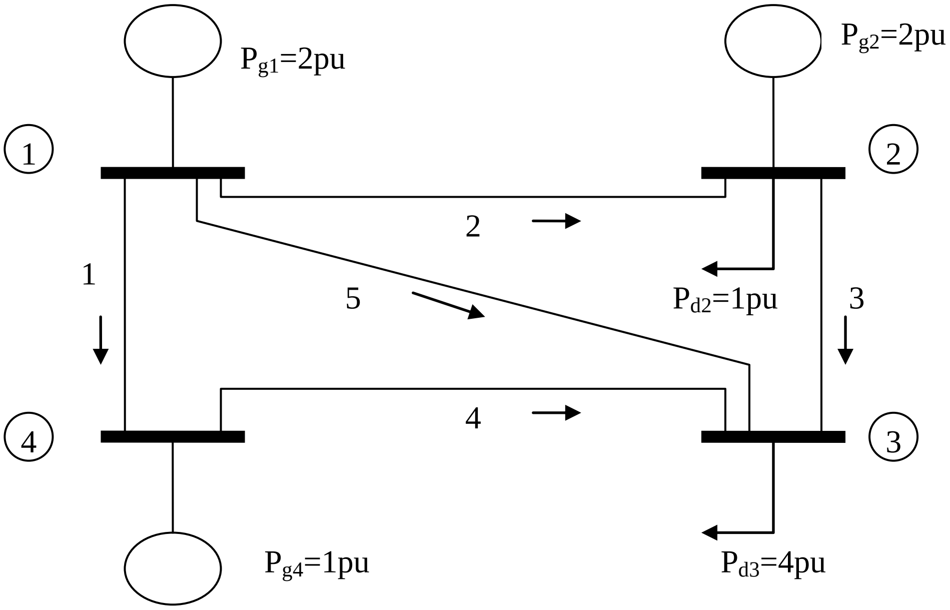
Therefore,

- X_{in} is the element of $\underline{X}$ in row i, column n.
- X_{jn} is the element of $\underline{X}$ in row j, column n.
- X_{im} is the element of $\underline{X}$ in row i, column m.
- X_{jm} is the element of $\underline{X}$ in row j, column m.
- X_{nm} is the element of $\underline{X}$ in row n, column m.
- X_{nn} is the element of $\underline{X}$ in row n, column n.
- X_{mm} is the element of $\underline{X}$ in row m, column m.

3.0 A computationally efficient method to obtain LODFs

A significant problem with W&W's method of obtaining the LODFs is that it requires $\underline{X}=(\underline{B}')^{-1}$, and if the system is very large, then inverting the matrix can be a computationally intense problem. We provide another method in this section. Our treatment is adapted from [1].

Let's reconsider our familiar 4-bus, 5 branch example problem.



The B' matrix for this system is

$$\underline{B}'=\begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}$$

What happens to B' if we lose the circuit #3 (from bus 2 to bus 3)?

We could re-develop the new B' from the one-line diagram as we are accustomed to doing now. Another way is to discern how the circuit #3 affects the B' matrix, in that it will affect exactly 4 elements, as indicated with the underlines below, corresponding to elements in bus numbered positions (2,2), (2,3), (3,2), and (3,3).

$$\underline{B}'=\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc|c} 2 & 3 & 4 & \\ \underline{20} & \underline{-10} & 0 & 2 \\ \underline{-10} & \underline{30} & -10 & 3 \\ 0 & -10 & 20 & 4 \end{array}$$

Recalling that all branch admittances of our network are –j10, what would these four elements be if branch #3 (between buses 2 and 3) were not there?

$$\underline{B}'^{out} = \begin{array}{c} \begin{matrix} 2 & \quad 3 & \quad 4 \end{matrix} \\ \begin{bmatrix} \underline{10} & \underline{0} & 0 \\ \underline{0} & \underline{20} & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

What is the difference between $\underline{B}'$ and $\underline{B}'^{out}$?

$$\Delta\underline{B}' = \underline{B}'^{out} - \underline{B}' = \begin{bmatrix} \underline{10} & \underline{0} & 0 \\ \underline{0} & \underline{20} & -10 \\ 0 & -10 & 20 \end{bmatrix} - \begin{bmatrix} \underline{20} & \underline{-10} & 0 \\ \underline{-10} & \underline{30} & -10 \\ 0 & -10 & 20 \end{bmatrix} = \begin{bmatrix} \underline{-10} & \underline{10} & 0 \\ \underline{10} & \underline{-10} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that the elements in $\Delta\underline{B}'$ are all multiples of $B'_{23}=-10$, i.e.,

$$\Delta\underline{B}' = -10\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that the above matrix can be expressed as

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

From this simple illustration, we can see a generalization, that whenever we remove a branch between buses i and j, with corresponding $\underline{B}'$ matrix element B'_{ij}, the $\underline{B}'$ matrix will change as indicated below.

$$\Delta\underline{B}' = \underline{B}'^{out} - \underline{B}' = b_{ij} \begin{array}{c} \begin{array}{ccccccc} & & i & & j & & \\ & & \downarrow & & \downarrow & & \end{array} \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \\ 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -1 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \qquad (22)$$

where b_{ij} is the susceptance of branch i-j. We use b_{ij} instead of B'_{ij} in order to ensure we have a defined term even when i or j are the swing bus. Notice that b_{ij} will always be negative.

The previous relation may be expressed as

$$\Delta\underline{B}' = b_{ij} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \times \begin{array}{c} \begin{bmatrix} 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \end{bmatrix} \\ \begin{array}{ccccccc} & & \uparrow & & \uparrow & & \\ & & i & & j & & \end{array} \end{array} \qquad (23)$$

If we define

$$\underline{e}_{ij} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \qquad (24)$$

then (23) becomes

$$\Delta\underline{B}' = b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^T \tag{25}$$

<u>Special case</u>: If the branch to be outaged is connected to the swing bus (in our case, it is bus #1), then,

if i=1,

$$\underline{e}_{1j} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow j \\ \\ \\ \end{array}$$

if j=1,

$$\underline{e}_{i1} = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \\ \\ \end{array}$$

From (22), and using (25), we have that

$$\Delta\underline{B}' = \underline{B}'^{out} - \underline{B}' = b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^T \tag{26}$$

Therefore the post-contingency B’ matrix can be expressed as

$$\underline{B}'^{out} = \underline{B}' + \Delta\underline{B}' = \underline{B}' + b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^T \tag{27}$$

From (1), we recall the DC power flow relation as

$$\underline{P} = \underline{B}'\,\underline{\theta} \tag{1}$$

If, when we remove the branch connected between buses i and j, the angles change by $\Delta\underline{\theta}$, then the new (post-contingency) angles will be $\underline{\theta}+\Delta\underline{\theta}$, and (1) becomes

$$\underline{P} = \underline{B}'^{out}\,(\underline{\theta} + \Delta\underline{\theta}) \tag{28}$$

Substituting (27) into (28), we obtain

$$\underline{P} = \left(\underline{B}' + b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^T\right)(\underline{\theta} + \Delta\underline{\theta}) \tag{29}$$

We can solve for the new angles according to

$$\underline{\theta} + \Delta\underline{\theta} = \left(\underline{B}' + b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1}\underline{P} \qquad (30)$$

We do not seem to have made much progress, because we still have to take an inverse…

However, there is a significant benefit to writing the new matrix in the way that we have written it, and that benefit becomes apparent if we learn a certain matrix relation. This relation is generally referred to as a lemma.

Matrix Inversion Lemma (MIL): Assume B’ is a nonsingular n×n matrix, and let $\underline{c}$ and $\underline{d}$ be n×M matrices with M<n. Then:

$$\left(\underline{B}' + \underline{c}\underline{d}^{T}\right)^{-1} = \underline{B}'^{-1} - \underline{B}'^{-1}\underline{c}\left[\underline{I}^{(M)} + \underline{d}^{T}\underline{B}'^{-1}\underline{c}\right]^{-1}\underline{d}^{T}\underline{B}'^{-1}$$

where $\underline{I}^{(M)}$ is the M×M identity matrix.

We neglect the proof but mention that it is proved in [1, p. 100] by simply multiplying the right-hand-side of MIL by the expression inside the brackets of the left-hand-side, and showing that the product is the n×n identity matrix.

We also mention that MIL is *derived* in [2, pp. 138-140].

It may not be very obvious at this point that MIL will help us, since we see 4 different inverses on the right-hand-side of MIL. Let’s apply MIL to (30) to see what happens.

Observing that we can define

$$\underline{c} = b_{ij}\underline{e}_{ij}$$
$$\underline{d}^{T} = \underline{e}_{ij}^{T} \qquad (31)$$

we can apply MIL according to

$$\left(\underline{B}'+b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\right)^{-1} =$$
$$\underline{B}'^{-1}-\underline{B}'^{-1}\,b_{ij}\,\underline{e}_{ij}\left[\underline{I}^{(M)}+\underline{e}_{ij}^{T}\,\underline{B}'^{-1}\,b_{ij}\,\underline{e}_{ij}\right]^{-1}\underline{e}_{ij}^{T}\,\underline{B}'^{-1} \tag{32}$$

One of the inverses on the right-hand-side can be addressed right away, however, by identifying the dimensionality of the expression inside the right-hand-side brackets, [$\underline{I}^{(M)}+\underline{d}^{T}\underline{B}'^{-1}\underline{c}$]. Recalling from the MIL that M is the number of columns in $\underline{c}$ and $\underline{d}$, and noting from (31) that in our case, $\underline{c}$ and $\underline{d}$ have only M=1 column, we see that what is inside the right-hand-side brackets is a scalar quantity! So that inverse we can take, and accordingly, we can express (32) as:

$$\left(\underline{B}'+b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\right)^{-1}=\underline{B}'^{-1}-\frac{\underline{B}'^{-1}\,b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\,\underline{B}'^{-1}}{1+\underline{e}_{ij}^{T}\,\underline{B}'^{-1}\,b_{ij}\,\underline{e}_{ij}} \tag{33}$$

Pulling out the scalar multiplier b_{ij} from where it appears in both the numerator and denominator, we have

$$\left(\underline{B}'+b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\right)^{-1}=\underline{B}'^{-1}-\frac{b_{ij}\,\underline{B}'^{-1}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\,\underline{B}'^{-1}}{1+b_{ij}\,\underline{e}_{ij}^{T}\,\underline{B}'^{-1}\,\underline{e}_{ij}} \tag{34}$$

Now we can isolate b_{ij} to only one appearance in the expression by dividing top and bottom by it, resulting in:

$$\left(\underline{B}'+b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\right)^{-1}=\underline{B}'^{-1}-\frac{\underline{B}'^{-1}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\,\underline{B}'^{-1}}{\frac{1}{b_{ij}}+\underline{e}_{ij}^{T}\,\underline{B}'^{-1}\,\underline{e}_{ij}} \tag{35}$$

What we have just expressed in (35) is the right-hand-side of (30), repeated below for convenience:

$$\underline{\theta}+\Delta\underline{\theta}=\left(\underline{B}'+b_{ij}\,\underline{e}_{ij}\,\underline{e}_{ij}^{T}\right)^{-1}\underline{P} \tag{30}$$

Substituting (35) into (30), we obtain:

$$\underline{\theta}+\Delta\underline{\theta}=\left(\underline{B}'^{-1}-\frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}}{\frac{1}{b_{ij}}+\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}}\right)\underline{P} \tag{36}$$

Distributing the injection vector P̲ results in

$$\underline{\theta}+\Delta\underline{\theta}=\underline{B}'^{-1}\underline{P}-\frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{P}}{\frac{1}{b_{ij}}+\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \tag{37}$$

But $\underline{\theta}=\underline{B}'^{-1}\underline{P}$, and therefore we can replace the corresponding expressions in both right-hand-side terms to obtain:

$$\underline{\theta}+\Delta\underline{\theta}=\underline{\theta}-\frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{\theta}}{\frac{1}{b_{ij}}+\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \tag{38}$$

We can simplify a little more by investigating $\underline{e}_{ij}^{T}\underline{\theta}$ in the numerator. This would be:

$$\underline{e}_{ij}^{T}\underline{\theta}=\begin{bmatrix}0 & \cdots & \underset{\substack{\uparrow\\ i}}{1} & \cdots & \underset{\substack{\uparrow\\ j}}{-1} & \cdots & 0\end{bmatrix}\begin{bmatrix}\theta_2\\ \vdots\\ \theta_i\\ \vdots\\ \theta_j\\ \vdots\\ \theta_{n+1}\end{bmatrix}=\theta_i-\theta_j \tag{39}$$

Substituting (39) into (38) results in:

$$\underline{\theta}+\Delta\underline{\theta}=\underline{\theta}-\frac{\underline{B}'^{-1}\underline{e}_{ij}(\theta_i-\theta_j)}{\frac{1}{b_{ij}}+\underline{e}_{ij}^T\underline{B}'^{-1}\underline{e}_{ij}} \tag{40}$$

Now we have only two inverses left. Interestingly, they both pre-multiply $\underline{e}_{ij}$. That is, we observe that both inverses appear in $\underline{B}'^{-1}\underline{e}_{ij}$, an n×1 vector.

Question: Besides inverting $\underline{B}'^{-1}$, how might we evaluate this term?

When you don't know how to evaluate something, just name it. Then, if things don't get better right away, you can at least move on with a sort of indicator of where your problem lies.

So let's name this n×1 vector as g^{ij}, i.e.,

$$\underline{g}^{ij}=\underline{B}'^{-1}\underline{e}_{ij} \tag{41}$$

Not sure if that helps much but it does indicate that

$$\underline{B}'\underline{g}^{ij}=\underline{e}_{ij} \tag{42}$$

Equation (42) should stimulate a very good idea within your mind. Since we very well know $\underline{B}'$ and $\underline{e}_{ij}$, we can obtain g^{ij} through LU factorization. Doing so will give us everything we need to evaluate (40), which, when we substitute g^{ij} for $\underline{B}'^{-1}\underline{e}_{ij}$, becomes:

$$\underline{\theta}+\Delta\underline{\theta}=\underline{\theta}-\frac{(\theta_i-\theta_j)}{\frac{1}{b_{ij}}+\underline{e}_{ij}^T\underline{g}^{ij}}\underline{g}^{ij} \tag{43}$$

One last small change should be made to (43), and that is to recognize that the term in the denominator $\underline{e}_{ij}^T g^{ij}$ can be expressed as

$$\underline{e}_{ij}^T \underline{g}^{ij} = \begin{bmatrix} 0 & \cdots & \underset{\substack{\uparrow \\ i}}{1} & \cdots & \underset{\substack{\uparrow \\ j}}{-1} & \cdots & 0 \end{bmatrix} \begin{bmatrix} g_2^{ij} \\ \vdots \\ g_i^{ij} \\ \vdots \\ g_j^{ij} \\ \vdots \\ g_{n+1}^{ij} \end{bmatrix} = g_i^{ij} - g_j^{ij} \tag{44}$$

Therefore, (43) becomes

$$\underline{\theta} + \Delta\underline{\theta} = \underline{\theta} - \frac{(\theta_i - \theta_j)}{\frac{1}{b_{ij}} + (g_i^{ij} - g_j^{ij})} \underline{g}^{ij} \tag{45}$$

Now what is the LODF? Recall the definition of the LODF is

$$d_{\ell,k} = \Delta f_\ell \ / \ f_{k0} \tag{16}$$

where we recall that

- k designates the outaged circuit, terminated by buses i and j;
- ℓ designates the circuit for which we want to compute the new flow, terminated by buses m and n.

First, let’s express the denominator of (16) f_{k0}, which is

$$f_{k0} = -b_{ij}(\theta_i - \theta_j) = -b_{ij}\underline{e}_{ij}^T \underline{\theta} \tag{46}$$

Now let’s express the numerator of (16) Δf_ℓ, which is

$$\Delta f_\ell = -b_{mn}(\Delta\theta_m - \Delta\theta_n) = -b_{mn}\underline{e}_{mn}^T \Delta\underline{\theta} \tag{47}$$

But note that $\Delta\underline{\theta}$ in (47) can be expressed using the second term of (45), i.e.,

$$\Delta\underline{\theta} = -\frac{(\theta_i - \theta_j)}{\frac{1}{b_{ij}} + (g_i^{ij} - g_j^{ij})}\underline{g}^{ij} \tag{48}$$

Substituting (48) into (47) results in

$$\Delta f_{\ell} = b_{mn}\underline{e}_{mn}^{T}\frac{(\theta_i - \theta_j)}{\frac{1}{b_{ij}} + (g_i^{ij} - g_j^{ij})}\underline{g}^{ij} \tag{49}$$

It is of interest at this point to rearrange (49) according to

$$\Delta f_{\ell} = b_{mn}\frac{b_{ij}(\theta_i - \theta_j)}{1 + b_{ij}(g_i^{ij} - g_j^{ij})}\underline{e}_{mn}^{T}\underline{g}^{ij} \tag{50}$$

We recognize in (50) that

$$f_{k0} = -b_{ij}(\theta_i - \theta_j) \tag{51}$$

and

$$\underline{e}_{mn}^{T}\underline{g}^{ij} = g_m^{ij} - g_n^{ij} \tag{52}$$

Substituting (51) and (52) into (50) results in

$$\Delta f_{\ell} = -b_{mn}\frac{f_{k0}(g_m^{ij} - g_n^{ij})}{1 + b_{ij}(g_i^{ij} - g_j^{ij})} \tag{53}$$

So (53) can be used to obtain the change in flow on circuit ℓ (terminated by buses m and n) due to outage of circuit k (terminated by buses i and j).

To get the LODF, we divide (53) by f_{k0}, resulting in

$$d_{\ell,k} = \frac{\Delta f_{\ell}}{f_{k0}} = -b_{mn}\,\frac{(g_m^{ij} - g_n^{ij})}{1 + b_{ij}(g_i^{ij} - g_j^{ij})} \tag{54}$$

The approach, then, to using (54), is to factorize $\underline{B}$’ into the $\underline{L}$ and $\underline{U}$ factors *once*. Then, for each contingency k=1,…,N_C, we use forward and backwards substation to obtain the vector g^{ij}. The LODFs for every branch ℓ (terminated by buses m and n), and then computed from (54).

References:

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[1] A. Debs, “Modern Power Systems Control and Operation,” Kluwer, 1988.

[2] A. Monticelli, “State estimation in electric power systems, a generalized approach,” Kluwer, 1999.