Zbus

1.0 Introduction

The Zbus is the inverse of the Ybus, i.e.,

$$\underline{Z} = \underline{Y}^{-1} \tag{1}$$

Since we know that

$$\underline{I} = \underline{Y}\underline{V} \tag{2}$$

and therefore

$$\underline{V} = \underline{Y}^{-1}\underline{I} \tag{3}$$

then

$$\underline{V} = \underline{Z}\underline{I} \tag{4}$$

So Zbus relates the nodal current injections to the nodal voltages, as seen in (4).

In developing the power flow problem, we choose to work with Ybus. The reason for this is that the power flow problem requires an iterative solution that can be made very efficient when we use Ybus, due to the sparsity (lots of zeros) in the matrix used in performing the iteration (the Jacobian matrix – we will discuss this more later).

However, in developing fault analysis methods (done in EE 457), we will choose to work with Zbus. The main reason for choosing to work with Zbus in fault analysis is that, as we will see, Zbus quantities characterize conditions when all current injections are zero except one, corresponding to the faulted bus. We can use some creative thinking to express that one current injection (the fault current). Once we have that one current injection, eq. (4) is very easy to evaluate to obtain all bus voltages in the network, and once we have bus voltages, we can get all currents everywhere. These currents are the currents under the fault conditions and are used to design protection systems.

The Zbus is not sparse (no zeros). But fortunately, fault analysis does not require iterative solutions, and so computational benefit of sparsity is not significant in fault analysis.

2.0 The meaning of Zbus elements

We can write the Z-bus relation for the same network as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \qquad (11)$$

We understand by eq. (11) that the independent sources are all current sources, and eq. (11) allows us to compute the voltages resulting from those current sources being injected into the network. These current sources are the equivalent representation of the generator voltage sources.

Let's inspect more closely one of the equations in (11). Arbitrarily choose the second equation.

$$V_2 = Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3 \qquad (12)$$

Now solve for *driving point impedance* of bus 2, Z_{22}.

$$Z_{22} = \frac{V_2 - Z_{21}I_1 - Z_{23}I_3}{I_2} \qquad (13)$$

But what if we set I_1 and I_3 to 0, i.e., what if we open-circuit buses 1 and 3? In other words, let's idle all sources.

This means that we will open any current sources at nodes 1 and 3 so that there are no sources there (but there may be load impedances). Then eq. (13) is:

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=I_3=0} \qquad (14)$$

Equation (14) says that Z_{22} is the ratio of bus 2 voltage to the bus 2 current injection when all sources are idled. This is the definition of the Thevenin impedance!!!!

Conclusion: The diagonal elements of the Z-bus are the Thevenin equivalent impedances seen looking into the network at that bus.

This is useful because, as seen in the last set of notes, the current into the fault may be computed as

$$I_f'' = \frac{V_f}{Z_f + Z_{Thev}}$$

The remainder of these notes comes from chapter 9 of Bergen & Vittal. I often cover it in EE 456 (but other instructors do not). They show you how to construct the Z-bus for a large network without performing matrix inversion.

One important attribute to building the Z-bus for fault analysis is that the generator subtransient reactances should be included. This means that it is necessary to include an additional bus for every generator in order to enable distinguishing between the high side of the generator internal voltage from the

network side of the generator subtransient reactance. The figure below illustrates the difference between generator representation for power flow analysis (the focus of EE 456) and generator representation for fault analysis (the focus of EE 457).

Generator representation for power flow analysis

Generator representation for fault analysis
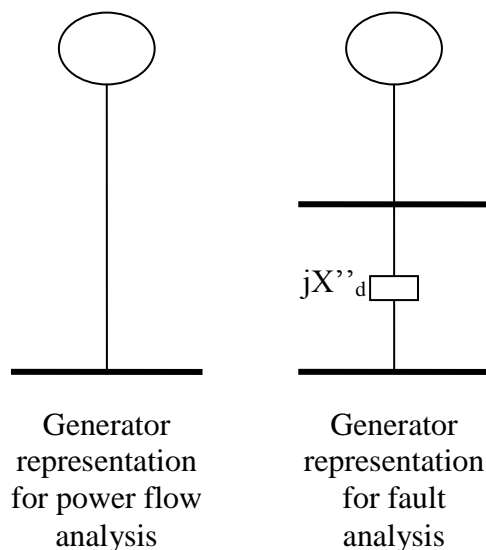
3.0 Self admittance and driving point impedance

You should recall that it is easy to develop the Y-bus. From that, one can invert it to obtain the Z-bus. However, in spite of the fact that Matlab is quite capable of matrix inversion for small dimension, you are NOT

ALLOWED to think about just inverting Ybus since we must, eventually, live in the real world of 5000+ bus models.

As a result, we must turn to the so-called Z-bus building algorithm, based on modification to an already existing Zbus. To understand it, we will first looking at modification to an already existing Ybus.

4.0 Ybus modifications (See Appendix 7)

Let's assume that we have a 3-bus system with Ybus given by

$$\underline{Y} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \tag{17}$$

Assume the branch between buses 1 and 3 is numbered as branch 3. Then let's modify branch 3 by adding another circuit between bus 1 and bus 3 having an admittance of Δy_3. How will the Ybus change?

- Add Δy_3 to diagonal elements in positions (1,1) and (3,3).
- Subtract Δy_3 from off-diagonal elements in positions (1,3) and (3,1).

The resulting matrix is

$$\underline{Y}^n = \begin{bmatrix} Y_{11} + \Delta y_3 & Y_{12} & Y_{13} - \Delta y_3 \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} - \Delta y_3 & Y_{32} & Y_{33} + \Delta y_3 \end{bmatrix} \qquad (18)$$

So the new Y-bus is just the old Y-bus with the addition of Δy_3 in four positions, as indicated by eq. (19):

$$\underline{Y}^n = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} + \begin{bmatrix} \Delta y_3 & 0 & -\Delta y_3 \\ 0 & 0 & 0 \\ -\Delta y_3 & 0 & \Delta y_3 \end{bmatrix}$$

$$= \underline{Y} + \Delta y_3 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad (19)$$

The matrix on the right of eq. (19) can actually be written as a product of two vectors. Define a vector corresponding to

the modification of branch 3 (connected between bus 1 and bus 3) as

$$a_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{matrix} \leftarrow bus1 \\ \\ \leftarrow bus3 \end{matrix} \quad \left(\text{or} \quad a_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow bus1 \\ \\ \leftarrow bus3 \end{matrix} \right) \quad (20)$$

Then notice that:

$$a_3 a_3^T = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad (21)$$

Substitution of (21) into (19) yields:

$$\underline{Y}^n = \underline{Y} + \Delta y_3 \underline{a}_3 \underline{a}_3^T \quad (22)$$

In general, anytime we modify Y-bus, then

$$\underline{Y}^n = \underline{Y} + \Delta y_k \underline{a}_k \underline{a}_k^T \quad (23)$$

where a_k is constructed according to:
- 1 at position i, -1 at position j if we add or remove a branch between buses i and j.
- 1 at bus i if we add or remove a shunt at bus i.

Why is eq. (23) of interest to us?

Reason is that there is a nice way to invert an expression in the form of eq. (23).

5.0 Matrix inversion lemma (Appendix 6)

The matrix inversion lemma (MIL), otherwise known as the Sherman-Morrison formula, is as follows.

Suppose we have an $n \times n$ symmetric matrix $\underline{Y}$ whose inverse is known and we wish to find the inverse of $\underline{Y} + \mu \underline{a}_k \underline{a}_k^T$, where

- μ is a scalar and
- $\underline{a}_k$ is an $n \times 1$ vector.

Then the MIL says that:

$$\left[\underline{Y} + \mu \underline{a}_k \underline{a}_k^T \right]^{-1} = \underline{Y}^{-1} - \gamma \underline{b}_k \underline{b}_k^T \qquad (24)$$

where $\underline{b}_k$ is an $n \times 1$ vector given by

$$\underline{b}_k = \underline{Y}^{-1} \underline{a}_k \qquad (25)$$

and γ is a scalar given by

$$\gamma = \left(\mu^{-1} + \underline{a}_k^T \underline{b}_k \right)^{-1} \qquad (26)$$

There is a proof of MIL in the text, page 600. It is also discussed in most books on linear algebra.

This formula is useful for getting a modified Z-bus, since $\underline{Z}=\underline{Y}^{-1}$. In other words, eq. (24), (25), and (26) become:

$$\underline{Z}^n = \left[\underline{Y} + \mu\underline{a}_k\underline{a}_k^T\right]^{-1} = \underline{Z} - \gamma\underline{b}_k\underline{b}_k^T \qquad (27)$$

where $\underline{b}_k$ is an n×1 vector given by

$$\underline{b}_k = \underline{Z}\underline{a}_k \qquad (28)$$

and γ is a scalar given by

$$\gamma = \left(\mu^{-1} + \underline{a}_k^T\underline{b}_k\right)^{-1} \qquad (29)$$

The implication is huge.

Consider that somehow, we are able to get an initial Zbus, denoted $\underline{Z}$, for some sub-portion of our network. Then we can change Zbus to reflect the addition of an element to the network, by using eqs. (28) and (29) to compute the $\underline{b}_k$ and γ, which are in turn used in eq. (27) to get the new Z-bus, denoted $\underline{Z}^n$.

Repeated application of this will eventually provide us with the entire Z-bus. The algorithm implementing this approach is called the Z-bus building algorithm.

6.0 Example (Example A7.1 in text)
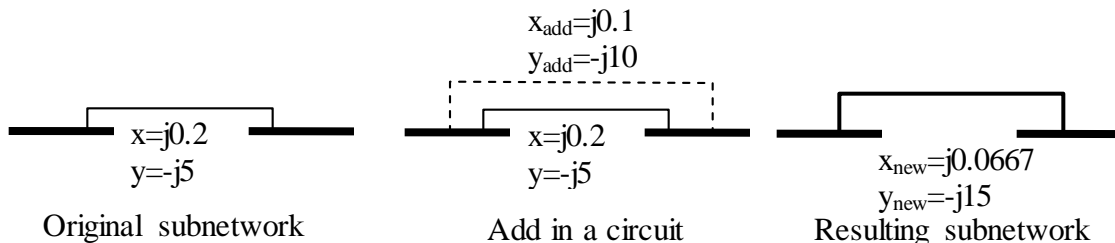
We are given the following Zbus for a 3-node network.

$$\underline{Z} = \frac{j}{3} \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$$

The admittance of branch 2, located between nodes 1 and 2, is changed from –j5 to –j15. Find the new Zbus, $\underline{Z}^n$.

Solution: The change in admittance of branch 2, located between buses 1 and 2, is

$$\Delta y_2 = \Delta y_{1,2} = -j10$$

Note that the fact that the admittance became more negative means that the impedance got less positive, i.e., the impedance decreased. This is caused by the addition of another circuit, as shown below.

$x_{add}=j0.1$
$y_{add}=-j10$

$x=j0.2$
$y=-j5$

Original subnetwork

$x=j0.2$
$y=-j5$

Add in a circuit

$x_{new}=j0.0667$
$y_{new}=-j15$

Resulting subnetwork

Following the example of eq. (20), we have:

$$a_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow bus1 \\ \leftarrow bus2 \\ \end{matrix}$$

Writing out eq. (23), we have:

$$\underline{Y}^n = \underline{Y} + \Delta y_k \underline{a}_k \underline{a}_k^T$$

$$= \underline{Y} + \Delta y_2 \underline{a}_2 \underline{a}_2^T = \underline{Y} + -j10 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

Now, if we had $\underline{Y}$, we could easily compute $\underline{Y}^n$. But that is not our objective. Our objective is to compute $\underline{Z}^n = (\underline{Y}^n)^{-1}$. By eq. (27), (28), and (29), this is

$$\underline{Z}^n = \left[\underline{Y} + \mu \underline{a}_2 \underline{a}_2^T \right]^{-1} = \underline{Z} - \gamma \underline{b}_2 \underline{b}_2^T$$

$$\underline{b}_2 = \underline{Z} \underline{a}_2$$

$$\gamma = \left(\mu^{-1} + \underline{a}_2^T \underline{b}_2 \right)^{-1}$$

where $\mu = \Delta y_2$.

So first let's compute $\underline{b}_2$. That is

$$b_2 = Za_2 = \frac{j}{3}\begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{j}{3}\begin{bmatrix} 0.1 \\ -0.4 \\ -0.1 \end{bmatrix}$$

What is $\underline{b}_2$? More generically, we observe:

$$b_2 = Za_2 = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} - Z_{12} \\ Z_{21} - Z_{22} \\ Z_{31} - Z_{32} \end{bmatrix}$$

In other words, $\underline{b}_2$ finds the difference between the columns of $\underline{Z}$ corresponding to the buses terminating the branch being modified, in this case, buses 1 and 2, i.e.,

$\underline{b}_2 = \underline{Z}_1 - \underline{Z}_2$, where $\underline{Z} = \begin{bmatrix} \underline{Z}_1 & \underline{Z}_2 & \underline{Z}_3 \end{bmatrix}$.
Now let's compute γ.

$$\gamma = \left(\mu^{-1} + \underline{a}_2^T \underline{b}_2\right)^{-1} = \left(\Delta y_2^{-1} + \underline{a}_2^T \underline{b}_2\right)^{-1}$$

$$= \left((-j10)^{-1} + \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\frac{j}{3}\begin{bmatrix} 0.1 \\ -0.4 \\ -0.1 \end{bmatrix}\right)^{-1}$$

$$= \left(j0.1 + \frac{j}{3}0.5\right)^{-1} = (j0.2667)^{-1} = -j3.75$$

Finally, we can compute $\underline{Z}^n$ according to:

$$\underline{Z}^n = \left[\underline{Y} + \mu\underline{a}_2\underline{a}_2^T\right]^{-1} = \underline{Z} - \gamma\underline{b}_2\underline{b}_2^T$$

$$= \frac{j}{3}\begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$$

$$-(-j3.75)\frac{j}{3}\begin{bmatrix} 0.1 \\ -0.4 \\ -0.1 \end{bmatrix}\frac{j}{3}\begin{bmatrix} 0.1 & -0.4 & -0.1 \end{bmatrix}$$

$$= \frac{j}{3}\begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} - \frac{j3.75}{9}\begin{bmatrix} 0.01 & -0.04 & -0.01 \\ -0.04 & 0.16 & 0.04 \\ -0.01 & 0.04 & 0.01 \end{bmatrix}$$

$$\underline{Z}^n = \frac{j}{3}\begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} - \frac{j1.25}{3}\begin{bmatrix} 0.01 & -0.04 & -0.01 \\ -0.04 & 0.16 & 0.04 \\ -0.01 & 0.04 & 0.01 \end{bmatrix}$$

$$= \frac{j}{3}\left(\begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} - \begin{bmatrix} 0.0125 & -0.05 & -0.0125 \\ -0.05 & 0.2 & 0.05 \\ -0.0125 & 0.05 & 0.0125 \end{bmatrix}\right)$$

$$= \frac{j}{3}\begin{bmatrix} 0.1875 & 0.15 & 0.1125 \\ 0.15 & 0.3 & 0.15 \\ 0.1125 & 0.15 & 0.1875 \end{bmatrix}$$

7.0 A closer look at γ

Recall that

$$\gamma = \left(\Delta y_b^{-1} + \underline{a}_k^T \underline{b}_k\right)^{-1}$$

But $\underline{b}_k = \underline{Z}\underline{a}_k$. Substitution yields:

$$\gamma = \left(\Delta y_b^{-1} + \underline{a}_k^T \underline{Z}\underline{a}_k\right)^{-1}$$

Consider the term $\underline{a}_k^T \underline{Z}\underline{a}_k$ where branch k connects buses i and j. Then we can write:

$\underline{a}_k^T \underline{Z}\underline{a}_k =$

$$[0 \quad \ldots \quad 1 \quad \ldots \quad -1 \quad \ldots \quad 0]
\begin{bmatrix}
Z_{11} & \ldots & Z_{1i} & \ldots & Z_{1j} & \ldots & Z_{1n} \\
\vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_{i1} & \ldots & Z_{ii} & \ldots & Z_{ij} & \ldots & Z_{in} \\
\vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_{j1} & \ldots & Z_{ji} & \ldots & Z_{jj} & \ldots & Z_{jn} \\
\vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z_{n1} & \ldots & Z_{ni} & \ldots & Z_{nj} & \ldots & Z_{nn}
\end{bmatrix}
\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix}
\begin{matrix} \\ \\ \leftarrow \text{bus i} \\ \\ \leftarrow \text{bus j} \\ \\ \\ \end{matrix}$$

$$= [Z_{i1} - Z_{j1} \quad \ldots \quad Z_{ii} - Z_{ji} \quad \ldots \quad Z_{ij} - Z_{jj} \quad \ldots \quad Z - Z_{jn}]
\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= Z_{ii} - Z_{ji} - (Z_{ij} - Z_{jj}) = Z_{ii} - Z_{ji} - Z_{ij} + Z_{jj} = Z_{ii} - 2Z_{ji} + Z_{jj}$$

$$\Rightarrow \underline{a}_k^T \underline{Z}\underline{a}_k = Z_{ii} - 2Z_{ji} + Z_{jj}$$

The expression for γ becomes

$$\gamma = \left(\Delta y_b^{-1} + \underline{a}_k^T \underline{Z} \underline{a}_k\right)^{-1} = \left(\Delta y_b^{-1} + Z_{ii} - 2Z_{ji} + Z_{jj}\right)^{-1}$$

If there is no circuit between buses i and j,

$$\gamma = \left(z_b + Z_{ii} - 2Z_{ji} + Z_{jj}\right)^{-1}$$

where $z_b = 1/\Delta y_b$ is the impedance of the new circuit added.

8.0 Some special cases

The MIL is used to build Zbus when you have two existing buses and you want to add in a line. But how do you get those existing buses in the first place? This requires some "special cases" for modifying Zbus. We will consider three "special cases," and then we will give the Z-bus building algorithm.

8.1 Adding a bus connected to ground

The situation is as illustrated in Fig. 5, where we are just adding in a bus with a shunt, but we are not (yet) connecting the bus to the network.
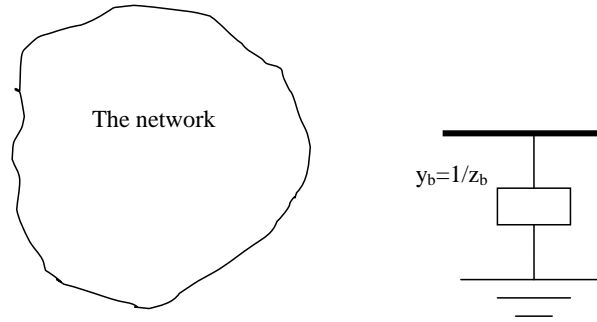
Fig. 5

Although this situation does not make much sense, it corresponds to a step that we must take in building the Z-bus.

To understand the approach to this situation, we return to the Ybus. How will Ybus be modified? We simply increase the dimension of Ybus by 1, with only the new diagonal element being non-zero, and it will have value y_b. The result of this is shown below, where Y is the Y-bus before the addition of the new node.

$$\underline{Y}^n = \left[\begin{array}{c|c} \underline{Y} & \underline{0} \\ \hline \underline{0}^T & y_b \end{array} \right] \tag{30}$$

Here, $\underline{0}$ is an n×1 vector of 0's. Inverting eq. (30), we get:

$$\underline{Z}^n = \left(\underline{Y}^n\right)^{-1} = \left[\begin{array}{c|c} \underline{Y} & \underline{0} \\ \hline \underline{0}^T & y_b \end{array} \right]^{-1} = \left[\begin{array}{c|c} \underline{Y}^{-1} & \underline{0} \\ \hline \underline{0}^T & 1/y_b \end{array} \right] = \left[\begin{array}{c|c} \underline{Z} & \underline{0} \\ \hline \underline{0}^T & z_b \end{array} \right] \tag{31}$$

So modifying Zbus to accommodate a new bus connected to ground is easy ☺, and it results in Modification #1 and Rule #1 in Section 9.5.

Modification #1: Add a branch with impedance z_b from a new bus (numbered n+1) to the reference (ground) node.

Rule #1: $\underline{Z}^n$ is given by

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{0} \\ \underline{0}^T & z_b \end{bmatrix} \qquad (32)$$

8.2 Add a branch from new to existing bus

We assume the new bus is numbered n+1 and the existing bus is numbered i.
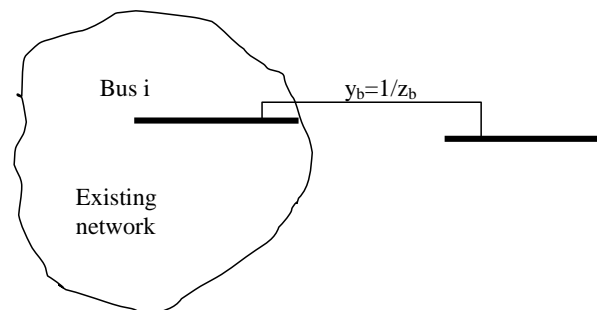The situation is as illustrated in Fig. 6.

Fig. 6

The derivation of what to do here is based on the same approach taken in section 8.1, i.e., we first see what happens to the Ybus, and then consider the inversion of the Ybus. In this case, the derivation is a little tedious and I will not go through it. You can refer to the text, pp. 604-605.

The result is Modification #2 and Rule #2 in Section 9.5.

Modification #2: Add a branch with impedance z_b from a new bus (numbered n+1) to an existing node i.

Rule #2: Denote the ith column of $\underline{Z}$ as $\underline{Z}_i$, and the ii^{th} element of $\underline{Z}$ as Z_{ii}. $\underline{Z}^n$ is then given by

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix} \qquad (33)$$

Example 1: A network consists of a single bus connected to the reference node through an impedance of j1.25 ohms. Give Zbus.

Solution: This requires application of rule 1.

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{0} \\ \underline{0}^T & z_b \end{bmatrix}$$

But $\underline{Z}$ does not exist, i.e., it is a matrix of dimension 0×0. Therefore,

$$\underline{Z}^n = \begin{bmatrix} j1.25 \end{bmatrix}$$

Example 2: A bus 2 is added to the network of example 1 through a branch having a reactance of j0.0533. Give new Zbus.

Solution: This requires application of rule 2.

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix}$$

Here, because $\underline{Z}$ consists of just a single bus (bus 1), $\underline{Z} = \underline{Z}_i = \underline{Z}_i^T = \underline{Z}_{ii}$, i.e., $\underline{Z} = \underline{Z}_1 = \underline{Z}_1^T = \underline{Z}_{11}$. Therefore

$$\underline{Z}^n = \begin{bmatrix} j1.25 & j1.25 \\ j1.25 & z_b + j1.25 \end{bmatrix} = \begin{bmatrix} j1.25 & j1.25 \\ j1.25 & j1.3033 \end{bmatrix}$$

8.3 Add a branch between existing buses

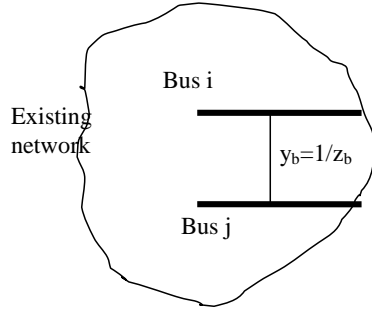This is the original situation we worked on to motivate MIL. It is illustrated in Fig. 8.

Fig. 8

We know what to do with this. The result is summarized as Modification #4 and Rule #4 in Section 9.5.

Modification # 4: Add a branch z_b between existing i^{th} and j^{th} nodes.

Rule #4: Denote the ith column of $\underline{Z}$ as $\underline{Z}_i$, and the jth column of $\underline{Z}$ as $\underline{Z}_j$, and denote the ii^{th}, jj^{th}, and ij^{th} elements of $\underline{Z}$ as Z_{ii}, Z_{jj}, Z_{ij}. Then $\underline{Z}^n$ is given by

$$\underline{Z}^n = \underline{Z} - \gamma \underline{b} \underline{b}^T$$

$$\underline{b} = \underline{Z}_i - \underline{Z}_j$$

$$\gamma = \left(z_b + Z_{ii} - 2Z_{ji} + Z_{jj} \right)^{-1}$$

We have already given one example of this rule (Example A7.1, page 12 above), and another one is given in the text as Example 9.11. So we will not illustrate further.

There is one more modification necessary, that we will call modification #3, and that is to add a branch with impedance Z_b between existing node and reference (ground). This differs from Modification #1 because in Modification #1, the node did not exist previously. Now it does. The issue is a bit tricky, and we will address it in section 8.5.

Before we do that, however, it may help to take a look at the algorithm used to build Z-bus, as we have enough information to do that now.

8.4 Z-bus building algorithm

Step 0: Number the nodes of the network starting with those nodes at the ends of branches connected to the reference node.

Step 1: Develop the Z-bus for all buses with a connection to the reference node. This is a Modification #1 using Rule #1, which means that the resulting matrix will be a diagonal matrix consisting of the values of the shunt impedances along the diagonal (all off-diagonals will be zero).

Recall that this step is based on:

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{0} \\ \underline{0}^T & z_b \end{bmatrix}$$

The network

$y_b = 1/z_b$
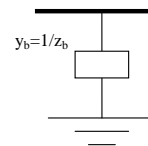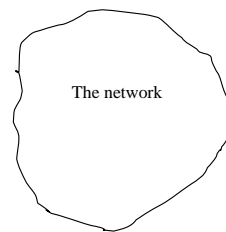
Note: building the Zbus in this way avoids the necessity of rule 3, since this step consists only of adding shunt impedances simultaneous with nodes (modification #1, rule #1), and after we are done, we will not have any more shunt impedances to add, and therefore it will not be possible to add a shunt impedance to an existing bus. Lovely.

Step 2: Add a new node to the ith (existing) node of the network via a new branch having impedance z_b. Continue until all nodes of the network have been added. This is modification #2, Rule #2, based on:

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix}$$
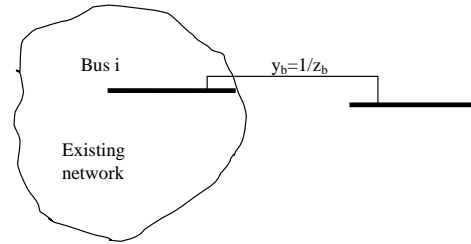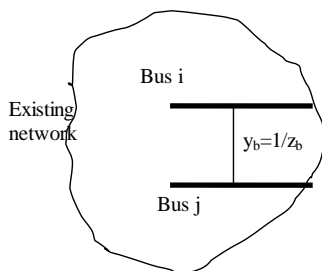

Step 3: Add a branch between the ith and j[th] nodes. Continue until all remaining lines have been added. This is modification #4, Rule #4, which is based on:

$$\underline{Z}^n = \underline{Z} - \gamma \underline{b}\underline{b}^T$$

$$\underline{b} = \underline{Z}_i - \underline{Z}_j$$

$$\gamma = \left(z_b + Z_{ii} - 2Z_{ji} + Z_{jj} \right)^{-1}$$

8.5 Adding shunt to existing bus: motivation

The Zbus building algorithm provides a way to build the Zbus using only Rules #1, 2, and 4. The reason why this works is that the first step of the algorithm is to build the Zbus for all buses that have shunt elements. This step eliminates the possibility that at some later step of the algorithm, we will have to add a shunt to an existing bus. Thus we avoid Rule #3, which addresses adding a shunt to an existing bus.

Unfortunately, there are situations that require Rule #3 which we cannot avoid. Consider, for example, that you want to develop a program which will compute fault currents for each bus, one faulted bus at a time.

An important question is, "What does *faulted bus* mean?"

It means that a short-circuit is placed on the bus. In reality, there is no perfect short, i.e., any short will always have some impedance. In effect, then, *faulted bus* means connecting a very small shunt impedance from a bus to the reference node.

So, assuming fault analysis requires the Zbus for each faulted condition, how would you do the study to get the short circuit current for each faulted bus?

Using our algorithm described above, what we have to do is to re-build the Zbus for each separate faulted bus we want to analyze.

Building Zbus is not as computational as inverting the matrix, but it does require some computational effort.

What we would like to do, from a programming perspective, is to build the network Zbus once, and then perform a more efficient manipulation for getting each fault-specific Zbus. This, then, is where Rule #3 comes in very handy.

Adding a shunt impedance from bus i to the reference node may be thought of in a preliminary way as adding a branch between an existing node i and a new node. This is a Rule #2 issue.

How do we handle Rule #2? Repeating eq. (33), we develop the new Zbus as:

$$\underline{Z}^n = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix} \qquad (33)$$

where the new Zbus relation is

$$\begin{bmatrix} \underline{V} \\ V_{ref} \end{bmatrix} = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix} \begin{bmatrix} \underline{I} \\ I_{ref} \end{bmatrix} \qquad (34)$$

But in eq. (34), $V_{ref}=0$, and so

$$\begin{bmatrix} \underline{V} \\ \hline 0 \end{bmatrix} = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix} \begin{bmatrix} \underline{I} \\ \hline I_{ref} \end{bmatrix} \qquad (35)$$

This fact that the left-hand-side of eq. (35) is 0 for the last equation is significant. It means that we may eliminate a variable from our solution vector. In particular, we would like to eliminate I_{ref} since it corresponds to a current that is not useful to us (it is the sum of all currents injected from the reference node into the network).

There exists a certain analytical technique for eliminating I_{ref}. It is called Kron Reduction after the famous power system engineer Gabriel Kron.

8.5 Kron Reduction (section 9.3)

Consider the following matrix equation:

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \qquad (36)$$

We can write this as separate equations:

$$x = ay + bz + e \qquad (37)$$
$$0 = cy + dz + f \qquad (38)$$

Let's eliminate the variable z from the top equation. This is accomplished by multiplying the bottom equation by $-bd^{-1}$

$$x = ay + bz + e$$

$$0 = -bd^{-1}cy - bd^{-1}dz - bd^{-1}f$$

and adding it to the top equation.

This results in

$$x = ay - bd^{-1}cy + bz - bd^{-1}dz + e - bd^{-1}f \quad (39)$$

Factoring out y from the first two terms and noting the third and fourth terms go to 0, we have:

$$x = (a - bd^{-1}c)y + e - bd^{-1}f \qquad (40)$$

Conclusion: we can eliminate the second variable (z) from our equation set if we force to zero the element in the first row, second column (b).

This works if the bottom element in the left-hand-side is zero. In this case, the operation to accomplish our purpose will not change the top element in the left-hand-side (x).

Let's see if we can do this same thing to eq. (35), repeated here for convenience:

$$\begin{bmatrix} \underline{V} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix} \begin{bmatrix} \underline{I} \\ I_{ref} \end{bmatrix} \qquad (35)$$

One can simply apply the same "pattern" that we used in the above case, which was:

$$x = (a - bd^{-1}c)y + e - bd^{-1}f \qquad (40)$$

to obtain (note that e and f are zero):

$$\underline{V} = \left(\underline{Z} - \underline{Z}_i (Z_{ii} + z_b)^{-1} \underline{Z}_i^T \right) \underline{I} \qquad (41)$$

Alternatively, one can think in terms of forcing the element in the second column, first equation, to zero, through an operation where we add a multiple of the second equation to the first.

The goal here is to zero out the element in position (1,2) of the eq. (35) matrix. The reason we want to do this is because then, the first equation becomes independent of the variable I_{ref}.

To do this, we will multiply the second row by $-\underline{Z}_i(Z_{ii}+z_b)^{-1}$. Then we add this multiplied second row to the first row. This results in:

$$\begin{bmatrix} \underline{V} - \underline{Z}_i(Z_{ii}+z_b)^{-1}*\mathbf{0} \\ \mathbf{0} \end{bmatrix} =$$

$$\begin{bmatrix} \underline{Z} - \underline{Z}_i(Z_{ii}+z_b)^{-1}*\underline{Z}_i^T & \underline{Z}_i - \underline{Z}_i(Z_{ii}+z_b)^{-1}(Z_{ii}+z_b) \\ \underline{Z}_i^T & Z_{ii}+z_b \end{bmatrix}\begin{bmatrix} \underline{I} \\ I_{ref} \end{bmatrix}$$

$$(42)$$

Noting in eq. (42) the terms that go to zero,

$$\begin{bmatrix} \underline{V} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \underline{Z} - \underline{Z}_i(Z_{ii}+z_b)^{-1}*\underline{Z}_i^T & \mathbf{0} \\ \underline{Z}_i^T & Z_{ii}+z_b \end{bmatrix}\begin{bmatrix} \underline{I} \\ I_{ref} \end{bmatrix}$$

$$(43)$$

And the first equation in (43) is independent of I_{ref}, so that we may extract it as:

$$\underline{V} = \left(\underline{Z} - \underline{Z}_i\left(Z_{ii}+z_b\right)^{-1}\underline{Z}_i^T\right)\underline{I}$$

$$(44)$$

which is the same as eq. (41).

This is Kron reduction. It can be applied to reduce the dimensionality of a set of linear equations whenever the left-hand-side of one of the equations is zero.

Aside: The text, in Section 9.3, motivates Kron reduction in a totally different, but equally legitimate way. It raises the issue of what you can do to the Ybus when you have a bus for which there is neither generation or load modeled at it. This issue comes often in power flow analysis. There are, in fact, some substations that do not have generation, and they do not serve load. Such substations might arise, for example, at the junction of several transmission circuits, or at a transmission-level transformer application (e.g., 169kV:345 kV). In this case, the left-hand side of the Ybus relation will be zero for the corresponding circuit. The text does some good examples of this on pp. 309-311.

8.6 Adding shunt to existing bus: method

We are now (finally) in position to state Modification #3 and Rule #3. It is:

Modification #3: Add a branch with impedance z_b between (existing) i^{th} node and the reference node.

Rule #3: Denote the ith column of $\underline{Z}$ as $\underline{Z}_i$, and the ii^{th} element of $\underline{Z}$ as Z_{ii}. $\underline{Z}^{n,e}$ is then given by

$$\underline{Z}^{n,e} = \begin{bmatrix} \underline{Z} & \underline{Z}_i \\ \underline{Z}_i^T & Z_{ii} + z_b \end{bmatrix} \qquad (33)$$

The new Zbus is then obtained by performing Kron reduction on $\underline{Z}^{n,e}$ to eliminate the last variable, that is,

$$\underline{Z}^n = \underline{Z} - \underline{Z}_i \left(Z_{ii} + z_b \right)^{-1} \underline{Z}_i^T \qquad (45)$$