EDC1

1.0 Introduction

In EE 303, we study the economic dispatch calculation (EDC) problem. We review our EE 303 work on EDC in this class, but we solve it in a different way. In addition, we extend the problem to account for losses.

Economic dispatch is the process of allocating the required load demand between the available generation units such that the cost of operation is at a minimum. The process of solving such a problem is referred to as optimization. Optimization problems are found in all engineering fields; in fact, some claim that engineering *is* optimization.

You have seen 1-dimensional optimization problems in calculus, when you found the minimum or maximum of a function. Generally, however, optimization problems are multi-variable.

An additional feature to general optimization problems is that they may be *constrained*. That is, we must find the minimum or maximum to a function subject to some kind of constraints on the variables of interest. The constraints may be either equality or inequalities.

A last very important feature of optimization problems is whether they are linear or nonlinear.

We will see that the EDC problem is a non-linear, multivariable, constrained optimization problem.

2.0 Optimization basics

Optimization is a decision-making tool. In light of this, we provide 2 basic definitions.

The *decision variables* are the variables in the problem, which, once known, determine the decision to be made.

The *objective function* is the function to be minimized or maximized. It is also sometimes known as the *cost function*.

The *constraints* are equality or inequality relations in terms of the decision variables which impose limitations or restrictions on what the solution may be, i.e., they *constrain* the solution.

Inequality constraints may be either *non-binding* or *binding*. A non-binding inequality constraint is one that does not influence the solution. A binding inequality constraint does restrict the solution, i.e., the objective function becomes “better” (greater if the problem is maximization or lesser if the problem is minimization) if a binding constraint is removed.

Optimization problems are often called *programs* or *programming* problems. Such terminology is not to be confused with use of the same terminology for a piece of source code (a program) or what you do when you write source code (programming). Use of the terminology here refers to an analytical statement of a decision problem. In fact, optimization problems are often referred to as *mathematical programs* and their solution procedures as *mathematical programming*. Such use of this terminology is indicated when one uses the term *linear programming* (LP), *nonlinear programming* (NLP), or *integer programming* (IP).

The general form of a nonlinear programming problem is to find vector $\underline{x}$ in:

$$\text{Min } f(\underline{x})$$

$$\text{subject to:} \qquad (1)$$

$$\underline{g}(\underline{x}) \leq \underline{b}$$

$$\underline{h}(\underline{x}) = \underline{c}$$

$$\text{and: } \underline{x} \geq 0$$

Here, f, g, and $\underline{h}$ are given functions of the n decision variables $\underline{x}$. The condition $\underline{x} \geq \underline{0}$ can be satisfied by appropriate definition of decision variables.

The LaGrangian function of (1) is:

$$F\left(\underline{x},\underline{\lambda},\underline{\mu}\right) \equiv f\left(\underline{x}\right) - \underline{\lambda}^{T}\left[\underline{h}\left(\underline{x}\right) - \underline{c}\right] - \underline{\mu}^{T}\left[\underline{g}\left(\underline{x}\right) - \underline{b}\right] \qquad (2)$$

where individual elements of $\underline{\lambda} = \left(\lambda_1, \lambda_2 \ldots, \lambda_m\right)$ and $\underline{\mu} = \left(\mu_1, \mu_2 \ldots, \mu_r\right)$ are called LaGrange multipliers.

The LaGrangian function is simply the summation of the objective function with the constraints. It is assumed that f, $\underline{h}$, and g are continuous and differentiable, that f is convex, and that the region in the space of decision variables defined by the inequality constraints is a convex region.

Given that $\underline{x}$ is a feasible point, the conditions for which the optimal solution occurs are:

$$\frac{\partial F}{\partial x_i} = 0 \qquad \forall\ i = 1, n \tag{3}$$

$$\frac{\partial F}{\partial \lambda_j} = 0 \qquad \forall\ j = 1, J \tag{4}$$

$$\mu_k \left[g_k(\underline{x}) - b_k \right] = 0 \qquad \forall\ k = 1, K \tag{5}$$

$$x_i \geq 0 \qquad \forall\ i = 1, n \tag{6}$$

These conditions are known as the Karush-Kuhn-Tucker (KKT) conditions or, more simply, as the Kuhn-Tucker (KT) conditions. The KKT conditions state that for an optimal point

1) The derivatives of the LaGrangian with respect to all decision variables must be zero (3).
2) All equality constraints must be satisfied (4).
3) A multiplier μ_k cannot be zero when its corresponding constraint is binding (5).
4) All decision variables must be non-negative at the optimum (6).

Requirement 3, corresponding to eq. (5), is called the "complementary" condition. The complementary condition is important to understand. It says that if $\underline{x}$ occurs on the boundary of the k^{th} inequality constraint, then $g_k(\underline{x}) = b_k$. In this case eq. (5) allows μ_k to be non-zero. Once it is known that the k^{th} constraint is binding, then the k^{th} constraint can be moved to the vector of equality constraints; i.e., $g_k(\underline{x})$ can then be renamed as $h_{m+1}(\underline{x})$ and μ_k as λ_{m+1}, according to:

$$\begin{aligned} g_k(\underline{x}) &\rightarrow h_{J+1}(\underline{x}) \\ \mu_k &\rightarrow \lambda_{J+1} \end{aligned} \qquad (7)$$

On the other hand, if the solution $\underline{x}$ does not occur on the boundary of the k^{th} inequality constraint, then (assuming $\underline{x}$ is an attainable point) $g_k(\underline{x}) - b_k < 0$. In this case, eq. (5) requires that $\mu_k = 0$ and the k^{th} constraint makes no contribution to the LaGrangian.

It is important to understand the significance of μ and λ. The optimal values of the LaGrangian Multipliers are in fact the rates of change of the optimum attainable objective value $f(\underline{x})$ with respect to changes in the right-hand-side elements of the constraints. Economists know these variables as shadow prices or marginal values. This information can be used not only to investigate changes to the original problem but also to accelerate repeat solutions. The marginal values λ_j or μ_k indicate how much the objective $f(\underline{x})$ would improve if a constraint b_j or c_k, respectively, were changed. One constraint often investigated for change is the maximum production of a plant.

3.0 EDC Problem Formulation

Each plant i has a cost-rate curve that gives the cost C_i in \$/hour as a function of its generation level P_{Gi} (the 3 phase power).

So we denote the cost-rate functions as $C_i(P_{Gi})$. These functions are normally assumed to be quadratic. For example, in Example 11.8 of the text, two such functions are given as

$$C_1(P_{G1}) = 900 + 45P_{G1} + 0.01P_{G1}^2 \tag{8}$$

$$C_1(P_{G2}) = 2500 + 43P_{G2} + 0.003P_{G2}^2 \tag{9}$$

If we have m generating units, then the total system cost will be given by

$$C_T = \sum_{i=1}^{m} C_i(P_{Gi}) \tag{10}$$

Equation (10), which corresponds to eq. (11.34) in the text, represents our objective function, and we desire to minimize it. The generation values P_{Gi} are the decision variables.

There are two basic kinds of constraints for our problem.

1. Power balance
2. Generation limits

3.1 Power balance constraint

In regards to power balance, it must be the case that the total generation equals the total demand P_D plus the total losses P_L.

$$\sum_{i=1}^{m} P_{Gi} = P_D + P_L \tag{11a}$$

The demand P_D is assumed to be a fixed value. However, the losses P_L depend on the solution (given by the P_{Gi}) which we do not know until we solve the problem. This dependency is due to the fact that the losses depend on the flows in the circuits, and the flows in the circuits depend on the generation dispatch. Therefore we represent this dependency according to eq. (11b).

$$\sum_{i=1}^{m} P_{Gi} = P_D + P_L(P_{G1}, P_{G2}, \ldots, P_{Gm}) \tag{11b}$$

One point made in the text (p.416) is that only *m-1* of the P_{Gi} are independent variables. Given the demand, one of the generation values, and the losses, are determined once the other m-1 of them are set.

In EE 303 (and in EE 456 if you took it), when we studied the power flow problem, this generator was referred to as the *swing bus*. We will assume this generator is unit 1.

Therefore we need to remove P_{G1} from the arguments of P_L so that eq. (11b) becomes

$$\sum_{i=1}^{m} P_{Gi} = P_D + P_L(P_{G2}, \ldots, P_{Gm}) \tag{11c}$$

We rearrange eq. (11c) so that all terms dependent on the decision variables are on the left-hand-side, according to:

$$\sum_{i=1}^{m} P_{Gi} - P_L(P_{G2}, \ldots, P_{Gm}) = P_D \tag{11d}$$

3.2 Generation limits

There are physical constraints on the generation levels. The generators cannot exceed their maximum capabilities, represented by $P_{Gi}^{\max}$. And clearly, they cannot operate below 0 (otherwise they are operating as a motor, attempting to drive the turbine – not a good operational state!). Most units actually cannot operate at 0; as a result, we will denote the minimum as $P_{Gi}^{\min}$. Therefore, the generation limits are represented by

$$P_{Gi}^{\min} \le P_{Gi} \le P_{Gi}^{\max} \qquad (12)$$

3.3 Problem statement

This leads us to the statement of the problem, i.e., the articulation of the mathematical program, which is, from eqs. (10), (11d), and (12), as follows.

Min $C_T = \sum_{i=1}^{m} C_i(P_{Gi})$

Subject to

$$\sum_{i=1}^{m} P_{Gi} - P_L(P_{G2}, \ldots, P_{Gm}) = P_D$$

$$P_{Gi}^{\min} \le P_{Gi} \le P_{Gi}^{\max} \qquad \forall \quad i = 1, \ldots, m$$

4.0 Application of KKT conditions

Recall the KKT conditions as given by eqs. (3-6). In formulating these conditions, we make one observation in regards to the complementary condition (5), which is repeated here for convenience.

$$\mu_k\left[g_k(\underline{x})-b_k\right]=0 \qquad \forall\, k=1,K \qquad (5)$$

This is a sort of "either-or" condition, i.e.,

- Either $u_k=0$ & $g_k(x)-b_k<0$ (non-binding) or
- $u_k\neq 0$ & $g_k(x)-b_k=0$ (binding)

However, we do not know in advance which it is.

So what we do is the following:

- Solve the problem without any inequality constraint
- Check solution against inequality constraints. For those that are violated, bring them in as equality constraints and re-solve the problem. Repeat this step until you obtain a solution for which no inequality constraints are violated.

This is an iterative solution procedure, and represents a procedural equivalent to the complementary condition. Thus, for any given iteration, we can assume there are no inequality constraints.

Therefore we may state the KKT conditions more simply as

$$\frac{\partial F}{\partial x_i} = 0 \qquad \forall\ i = 1, n \tag{3}$$

$$\frac{\partial F}{\partial \lambda_j} = 0 \qquad \forall\ j = 1, J \tag{4}$$

where it is assumed that any binding inequality constraints are included in eq. (4) as equality constraints.

Let's apply these conditions to the problem statement of Section 3.3 above, assuming that no inequality constraints are binding so that there is only one equality constraint to consider (the power balance constraint).

First, we form the Lagrangian function:

$$F = \sum_{i=1}^{m} C_i(P_{Gi}) - \lambda \left[\sum_{i=1}^{m} P_{Gi} - P_L(P_{G2}, \dots, P_{Gm}) - P_D \right] \quad (13)$$

Now applying the KKT conditions of (3) and (4), we get:

$$\frac{\partial F}{\partial P_{Gi}} = \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} - \lambda \left[1 - \frac{\partial P_L(P_{G2}, \dots, P_{Gm})}{\partial P_{Gi}} \right] = 0, \qquad \forall i = 1, \dots m \quad (14)$$

$$\frac{\partial F}{\partial \lambda} = \sum_{i=1}^{m} P_{Gi} - P_L(P_{G2}, \dots, P_{Gm}) - P_D = 0 \quad (15)$$

Observe that we have m equations of the form given in (14). However, the one corresponding to i=1 will not have a loss term and therefore will be:

$$\frac{\partial F}{\partial P_{G1}} = \frac{\partial C_1(P_{Gi1})}{\partial P_{G1}} - \lambda = 0 \quad (16)$$

Because $\frac{\partial P_L}{\partial P_1} = 0$, eq. (14) appropriately captures eq. (16). Nevertheless, your text specifies them separately (see eqs. (11.48-11.50) on page 417).

The term $\frac{\partial C_i}{\partial P_{Gi}}$ is called the incremental cost of unit i and is denoted by IC_i.

Let's consider eq. (14) more closely. In particular, let's solve it for λ.

$$\lambda = \frac{1}{\left[1 - \frac{\partial P_L(P_{G2}, \ldots, P_{Gm})}{\partial P_{Gi}}\right]} \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} \qquad \forall i = 1,\ldots m \tag{17}$$

Define the fraction out front as L_i, that is

$$L_i = \frac{1}{\left[1 - \frac{\partial P_L(P_{G2}, \ldots, P_{Gm})}{\partial P_{Gi}}\right]} \tag{18}$$

We call L_i the *penalty factor* for the i^{th} unit. Note that L_1=1.

Substituting eq. (18) into (17) results in

$$\lambda = L_i \frac{\partial C_i(P_{Gi})}{\partial P_{Gi}} \qquad \forall i = 1,\ldots m \tag{19}$$

What eq. (19) says is that, at the optimum dispatch, for each unit not at a binding inequality constraint, the product of the penalty factor and the incremental cost of unit is the same and is equal to λ.

Example:

We study ex. 11.8 in the text. In this example, the fuel-cost curves are given (and were referenced above in eqs. (8), (9)):

$$C_1(P_{G1}) = 900 + 45P_{G1} + 0.01P_{G1}^2 \qquad (8)$$

$$C_1(P_{G2}) = 2500 + 43P_{G2} + 0.003P_{G2}^2 \qquad (9)$$

The load is specified as P_D=700 MW, but let's solve it first for 600 MW. Generator limits are given as

$$50MW \le P_{G1} \le 200MW$$

$$50MW \le P_{G1} \le 600MW$$

In this example, we assume that there are no losses. This means that all penalty factors are 1.0. Assuming there are no binding inequality constraints, eq. (19) is

$$\lambda = \frac{\partial C_1(P_{G1})}{\partial P_{G1}} \qquad \lambda = \frac{\partial C_2(P_{G2})}{\partial P_{G2}}$$

Writing out these equations, we have:

$$\lambda = 45 + 0.02P_{G1}$$

$$\lambda = 43 + 0.006P_{G2}$$

We also have our equality constraint eq. (15)

$$P_{G1} + P_{G2} = 600$$

In EE 303, we learned to solve these equations in matrix representation, as a set of linear equations, as given below.

$$\begin{bmatrix} 0.02 & 0 & -1 \\ 0 & 0.006 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{G1} \\ P_{G2} \\ \lambda \end{bmatrix} = \begin{bmatrix} -45 \\ -43 \\ 600 \end{bmatrix}$$

Solution to this equation yields:

$$\begin{bmatrix} P_{G1} \\ P_{G2} \\ \lambda \end{bmatrix} = \begin{bmatrix} 61.64 \\ 538.46 \\ 46.23 \end{bmatrix}$$

Checking the inequality limits, we see that we have found the solution.

Let's explore another solution method. The previous one is fine, but it requires that all equations be linear. This may not always be the case, e.g., when we include losses, the power balance equation can be nonlinear. The method is known as Lambda-iteration and is best understood via Fig. 1 which shows incremental cost curves IC_1, IC_2, given by

$$IC_1 = 45 + 0.02P_{G1}$$

$$IC_2 = 43 + 0.006P_{G2}$$

The incremental cost curves are just the derivatives of the cost-rate curves. Observe that the expressions derived for λ under the KKT conditions specify a certain relation among the incremental cost curves. An implication here is that the incremental cost curves express the derivatives (or the incremental costs) under any condition, not necessarily at the optimum.

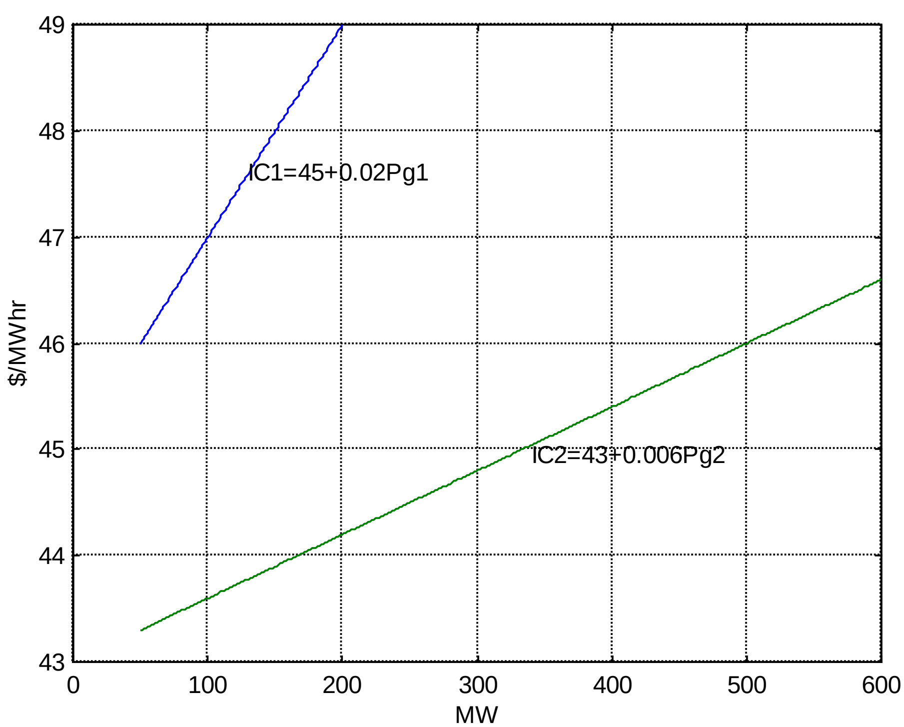


Fig. 1

The lambda iteration method begins with a guess in regards to a value of λ which satisfies the KKT conditions (such that all incremental costs are equal), and the total demand equals the load.

The lambda iteration may be performed graphically. Let’s guess that λ=46. To determine what the corresponding generation levels are at the optimum, draw a horizontal line across our IC curves, as shown by the dark horizontal line in Fig. 2.

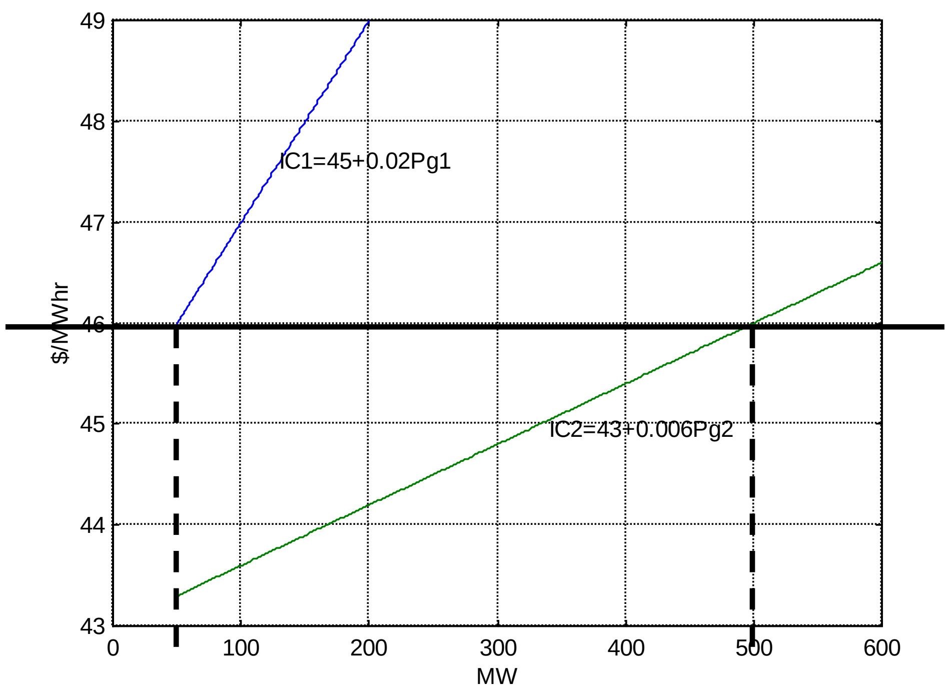


Fig. 2

The corresponding generation values are the dark vertical dashed lines, so we can see that P_{G1}=50 and P_{G2}=500, for a total generation of 550 MW. This is less than the desired 600 MW so let's increase our guess. Let's try about 46.4, as shown in Fig. 3.

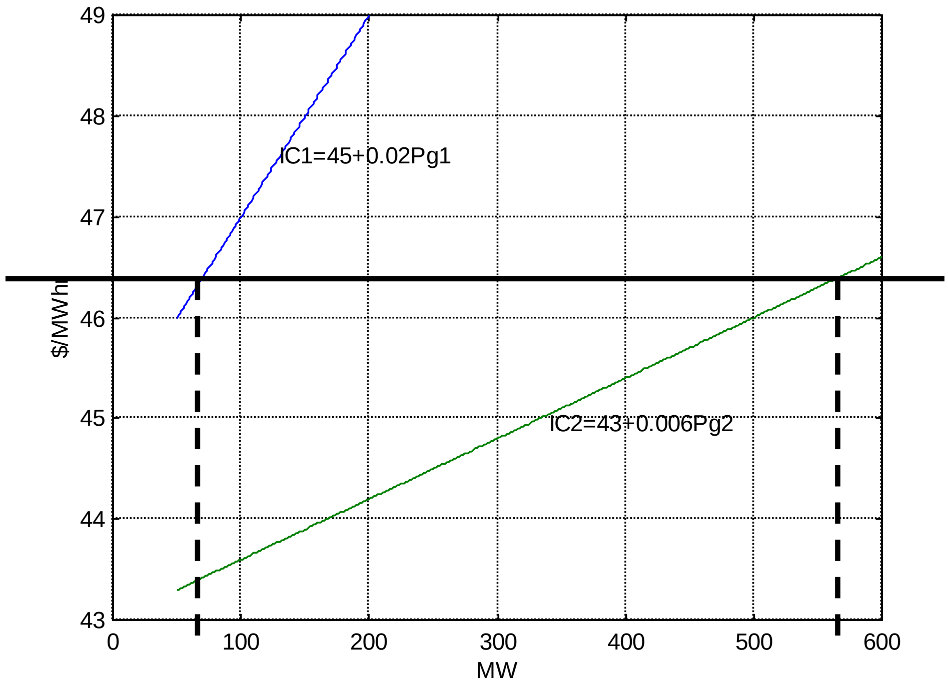


Fig. 3

The corresponding generation levels are about P_{G1}=65 MW and P_{G2}=565 MW, for a total of 630 MW, and so this is a little too high. Let's try λ=46.2 as shown in Fig. 4.

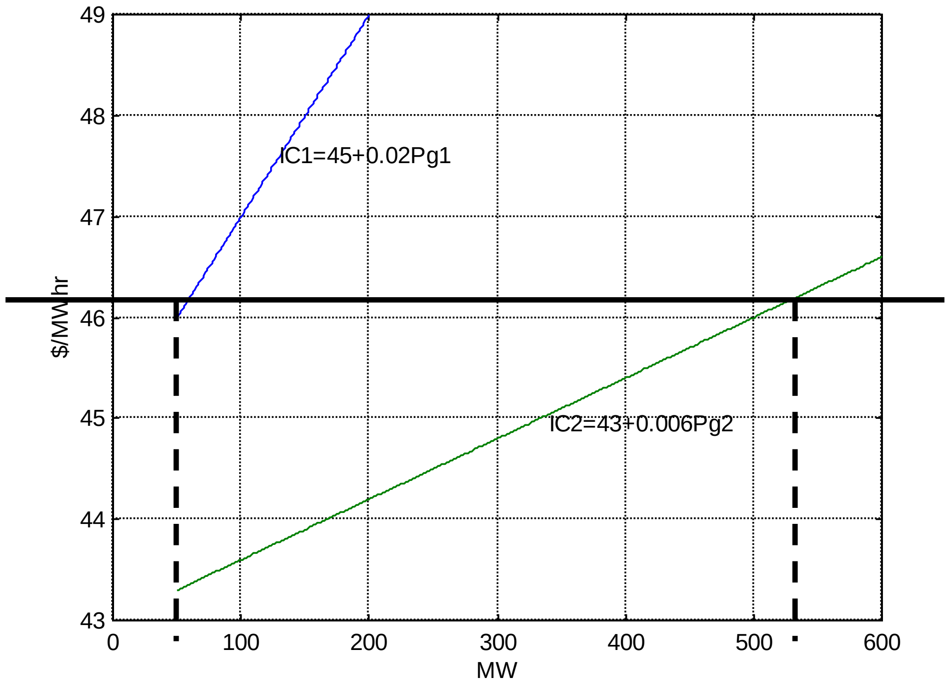


Fig. 4

The corresponding generation levels are about P_{G1}=55 MW and P_{G2}=540 MW for a total of 595 MW, so this is just a small bit too low. It is probably not possible to do better than this unless with use a more granular axis in our plots.

This method can be stated analytically as well. Notice what we are doing: we choose λ and then obtain the generation levels from the plots.

Well, the plots are really analytical relations between λ and the generation levels, and we can easily manipulate them so that they give the generation levels as a function of λ, as shown below.

$$0.02P_{G1} = \lambda - 45 \Rightarrow P_{G1} = 50\lambda - 2250$$

$$0.006P_{G2} = \lambda - 43 \Rightarrow P_{G2} = 166.67\lambda - 7166.7$$

Now we can proceed analytically.

Observe that $P_{G1}+P_{G2}=P_D=50\lambda-2250+166.67\lambda-7166.7$. Thus, we can solve for λ as a function of P_D. This may be helpful in Pr11.11.

As before, guess 46 and calculate:

$$P_{G1} = 50(46) - 2250 = 50$$

$$P_{G2} = 166.67(46) - 7166.7 = 500$$

Total is 550 MW which is too low so let's try 46.4 (we could try anything we like, as long as it is higher, since the generation is too low in our first guess):

$$P_{G1} = 50(46.4) - 2250 = 70$$

$$P_{G2} = 166.67(46.4) - 7166.7 = 566.78$$

Total is 636.78, so now we need to try a lower λ. But let's use linear interpolation to guide our next value of λ:

$$\frac{46.4-46}{636.78-550}=\frac{\lambda-46}{600-550}\Rightarrow\lambda=46.2305$$

Because our equations for P_{G1} and P_{G2} are linear with λ, the linear interpolation will provide an exact answer. We can check to see:

$$P_{G1}=50(46.2305)-2250=61.525$$

$$P_{G2}=166.67(46.2305)-7166.7=538.5374$$

And the sum is 600.06 MW, as desired.

This method will also work when the IC curves and/or the power balance equation are nonlinear. For nonlinear relations, however, linear interpolation will not find the solution in one shot, and so it is necessary to iterate. On each iteration, one may employ a stopping criterion by checking to see whether the total generation is within some tolerance of the load. This basic procedure is given on p.411 of the text.

<u>Example (extended)</u> (Ex 11.9 in text)
Now let's reconsider our example, but with a load of 700 MW instead of 600 MW. Using our graphical method again, and with the knowledge gained from our previous example, we know that λ will exceed 46.4. But it cannot go too much higher without causing Unit 2 to exceed its upper limit. Let's try it at the value of λ that causes unit 2 to be at its upper limit. This is shown in Fig. 5.

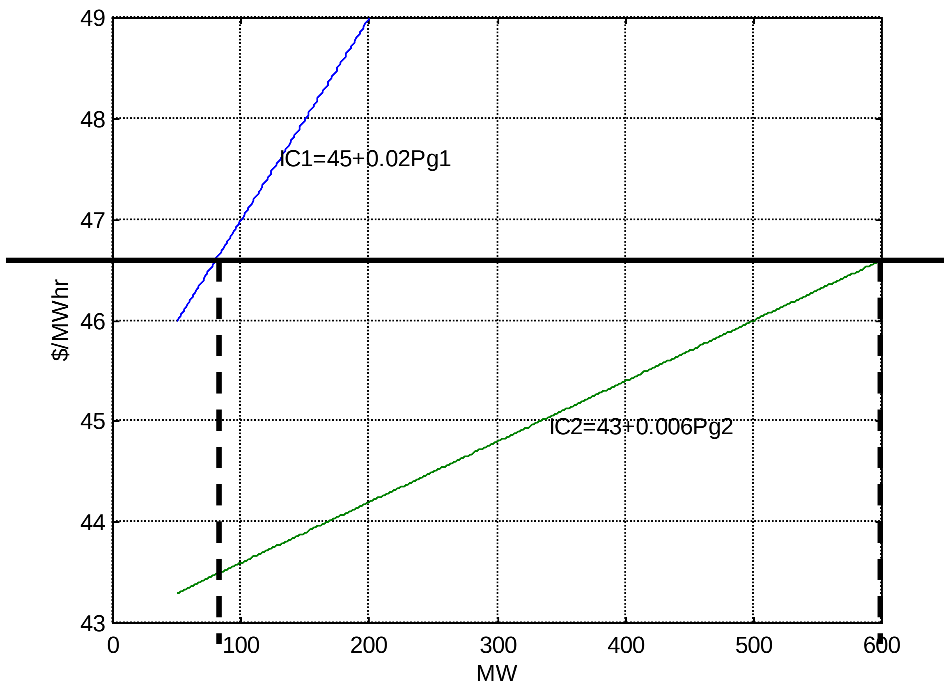


Fig. 5

It appears that λ is about 46.6, and $P_{G1} \approx 85$ MW, $P_{G2} \approx 600$ MW, for a total of 685 MW. So this is not enough, and we must therefore raise λ. However, we cannot raise λ on unit 2 because it is already at its upper limit. So we have to clamp P_{G2} at 600 MW. In other words, we will no longer use P_{G2} in our λ-iteration, although we will need to account for its generation of 600 MW.

So we will now perform λ-iteration on only the remaining units. In this case, the "remaining units" is just unit 1. In addition, our stopping criteria will now be that the total generation of the remaining units be equal to P_D-P_{g2}=700-600=100.

The upshot of this is that we need to perform λ-iteration on unit 1's ability to supply 100 MW. The horizontal solid-dark line of Fig. 6 illustrates.

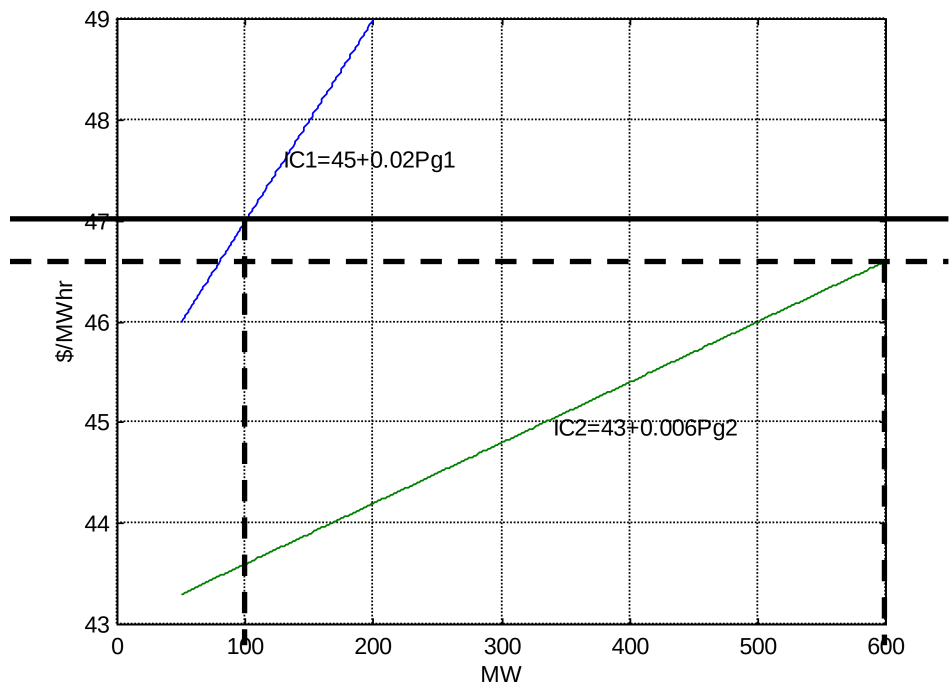


Fig. 5

Observe, however that there are now two horizontal lines,

- the solid one for unit 1 at 47;
- the dashed one for unit 2 at about 46.6

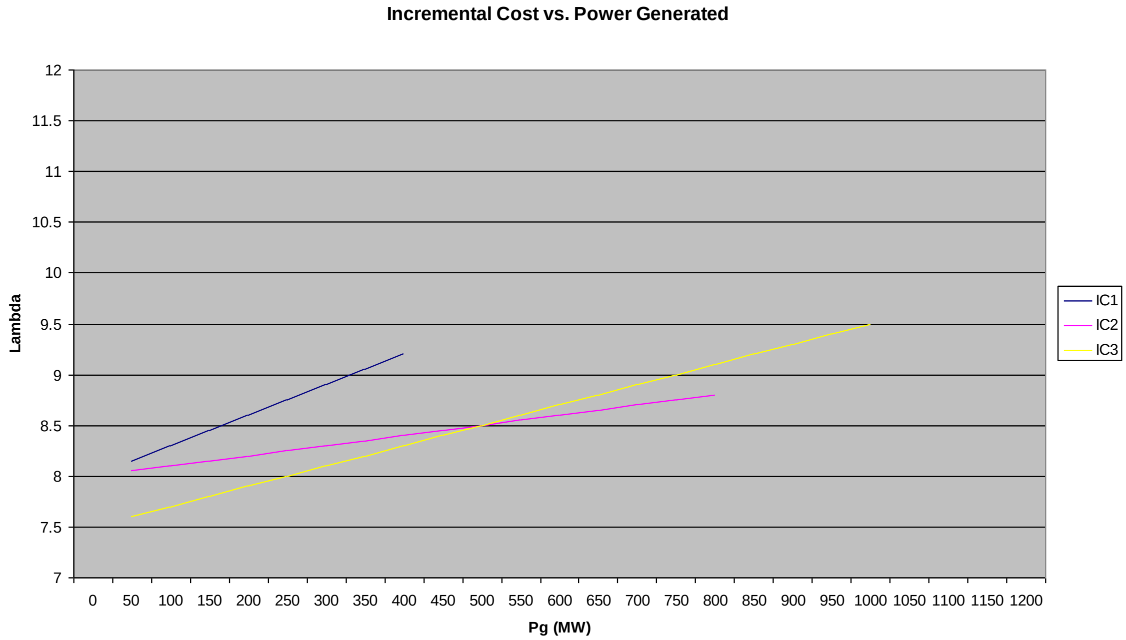
So which one is λ?

➔ λ is the SYSTEM incremental cost and indicates the cost of *optimally* supplying another MW from the system for the next hour. If the system has to supply another MW for the next hour, in this case (because there is only 1 regulating unit), it would have no choice but to do it with unit 1.

Therefore $\lambda=47$.

Then what is 46.6? It is the incremental cost of unit 2 (but not the system incremental cost). The unit incremental cost is normally understood as the cost for the unit to supply another MW for one hour. It can equivalently be understood as the savings if the unit was off-loaded by 1 MW for one hour, and in this case, that is a better interpretation since the unit cannot supply more power.

Hint on Problem 11.11: This problem is very similar to Example 11.10 in the text. And so study of Example 11.10 should help you a great deal in solving this problem. The other thing that will help you is the IC curves of the three units (which you should plot for problem 11.10). These IC curves are below.



The problem is asking you to assume that all three units are within a single power generating plant, and we want to obtain the composite PLANT incremental cost as a function of PLANT loading. To do this, observe:

- Min plant loading is $P_{G1,min}+P_{G2,min}+P_{G3,min}=50+50+50=$ 150MW, which can be obtained from the data of problem 11.10 or can be read off the above plots.
- Max plant loading is $P_{G1,max}+P_{G2,max}+P_{G3,max}=400+800+1000=$ 2200MW which can be obtained from the data of problem 11.10 or can be read off the above plots.

So you will obtain a set of piecewise linear curves from P_D=150 to P_D=2200, depending on which machines are regulating (not at their limits). You so this by finding the IC for each set of regulating conditions. For example, when IC=7.6, only unit 3 is regulating (units 1 and 2 are at their limits).