1.

(a) Find the α-level Neyman-Pearson rule (with false-alarm probability α) for testing

$$\begin{aligned} \mathcal{H}_0 &: \theta = \theta_0 \quad \text{versus} \\ \mathcal{H}_1 &: \theta = \theta_1. \end{aligned}$$

where

$$f_{X\,|\,\Theta}(x\,|\,\theta_0) = \begin{cases} c_0\, x^2, & |x| \le 1 \\ 0, & \text{else} \end{cases}$$

versus

$$f_{X\,|\,\Theta}(x\,|\,\theta_1) = \begin{cases} c_1\,(3 - |x|), & |x| \le 3 \\ 0, & \text{else} \end{cases}$$

and c_0 and c_1 are constants. (Determine c_0 and c_1.)

(b) Compute ROC $P_{\text{D}}(P_{\text{FA}})$ for this test.

2. Problem 3.13 in Kay-II.

3. We measure $X = x$, modeled as

$$X = \theta + W$$

where the noise W follows

$$W \sim \text{U}(-\tfrac{1}{2}, \tfrac{1}{2}).$$

(a) Find the α-level Neyman-Pearson rule (with false-alarm probability α) for testing

$$\begin{aligned} \mathcal{H}_0 :& \quad \theta = 0 \quad \text{versus} \\ \mathcal{H}_1 :& \quad \theta = \theta_1 \end{aligned}$$

where $0 < \theta_1 < 1$ is a known constant.

(b) Determine and sketch ROC for this test.

4. The measurement vector $\boldsymbol{X} = \boldsymbol{x} = [x[0], x[1]]^T$ follows the probabilistic model:

$$f_{X\,|\,a}(\boldsymbol{x}\,|\,a) = \mathcal{N}(\boldsymbol{x}\,|\,[a, 0]^T, C)$$

where a is an unknown parameter and C is the known covariance matrix:

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Consider the composite hypothesis-testing problem:

$$\begin{aligned} \mathcal{H}_0 &: \ a \in (0, 1) \quad \text{versus} \\ \mathcal{H}_1 &: \ a \in [1, +\infty). \end{aligned}$$

Show that a uniformly most powerful (UMP) test exists for this problem and find the UMP test of level $\alpha \in (0, 1)$. Provide all the details in your derivation.

5. The measurement $X = x$ follows the probabilistic model:

$$f_{X\,|\,\Theta}(x\,|\,\theta) = \text{Expon}(x\,|\,\theta) = \theta\ \exp(-\theta\, x)\, i_{(0,+\infty)}(x)$$

where θ is an unknown parameter.

(a) Consider the composite hypothesis-testing problem:

$$\begin{aligned} \mathcal{H}_0 &: \ \theta \in [1, 2) \quad \text{versus} \\ \mathcal{H}_1 &: \ \theta \in [2, +\infty). \end{aligned}$$

Show that UMP test exists for this problem and find the UMP test of level $\alpha \in (0, 1)$. Provide all the details in your derivation.

6. ***UMP testing with Laplacian observations.*** The measurement $X = x$ follows the probabilistic model:

$$f_{X\,|\,\Theta}(x\,|\,\theta) = \tfrac{1}{2}\,e^{-|x-\theta|}$$

where θ is an unknown parameter and $|\cdot|$ denotes the absolute value. Consider the composite hypothesis-testing problem:

$$\begin{aligned}\mathcal{H}_0 &: \quad \theta = 0 \quad \text{versus} \\ \mathcal{H}_1 &: \quad \theta \in (0, +\infty).\end{aligned}$$

Does UMP test exist? If so, find it and derive its ROC. If not, find the generalized likelihood ratio (GLR) test.

7. Problem 3 in Wasserman, Ch. 17.9.

8. Problem 7 in Wasserman, Ch. 17.9.