1. Problem 3.18 in Kay-II.

2. Find the Bayes' decision rule and the minimum preposterior (Bayes) risk for the simple binary hypothesis testing problem with equal priors $[\pi_0 = \pi_1 = \frac{1}{2}]$ and 0-1 loss, and

$$f_{X\,|\,\Theta}(x\,|\,\theta_0) = \begin{cases} c_0\, x^2, & |x| \le 1 \\ 0, & \text{else} \end{cases}$$

versus

$$f_{X\,|\,\Theta}(x\,|\,\theta_1) = \begin{cases} c_1\,(3 - |x|), & |x| \le 3 \\ 0, & \text{else} \end{cases}$$

where c_0 and c_1 are constants. (Determine c_0 and c_1.)

3. The measurement $X = x$ follows the probabilistic model:

$$f_{X\,|\,\Theta}(x\,|\,\theta) = \text{Expon}(x\,|\,\theta) = \theta\, \exp(-\theta\, x)\, i_{(0,+\infty)}(x)$$

where θ is an unknown parameter. We adopt the Bayesian approach with a uniform prior pdf for the unknown parameter θ:

$$\pi(\theta) = \text{U}(\theta\,|\,0,1) = i_{(0,1)}(\theta).$$

Find a Bayes' rule for testing

$$\mathcal{H}_0 \;:\; \Theta \in (0, 0.5) \quad \text{versus}$$
$$\mathcal{H}_1 \;:\; \Theta \in (0.5, 1).$$

Assume 0-1 loss.

4. The measurement vector $\boldsymbol{X} = \boldsymbol{x} = [x[0], x[1]]^T$ follows the probabilistic model:

$$f_{X \mid \mu}(\boldsymbol{x} \mid \boldsymbol{\mu}) = \mathcal{N}(\boldsymbol{x} \mid \boldsymbol{\mu}, C)$$

where $\boldsymbol{\mu}$ is an unknown parameter vector and C is the known covariance matrix:

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 + \rho^2 \end{bmatrix}$$

where $\rho \in (0, 1)$ is a known constant. Consider the following simple hypothesis-testing problem:

$$\begin{aligned} \mathcal{H}_0 &: \boldsymbol{\mu} = [-a, 0]^T \quad \text{versus} \\ \mathcal{H}_1 &: \boldsymbol{\mu} = [a, 0]^T \end{aligned}$$

and $a > 0$ is a known constant.

(a) Assuming equal priors ($\pi_0 = \pi_1 = \frac{1}{2}$), derive the detector that minimizes the average error probability. Show that the corresponding decision rule is given by

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & x[0] - b\,x[1] \geq \tau \\ 0, & x[0] - b\,x[1] < \tau \end{cases} .$$

(b) Determine the minimum average error probability achieved by this detector.

5. The signal Θ and noise W are independent random variables, where $\Theta \in \{-1, 1\}$ with prior probability mass function (pmf)

$$p_\Theta(-1) = p_\Theta(1) = \tfrac{1}{2}$$

and

$$W \sim \mathrm{U}(-2, 2).$$

We measure $X = x$, modeled as

$$X = \Theta + W.$$

(a) Find the MMSE estimate of Θ based on x.

(b) Derive the decision rule for testing

$$\mathcal{H}_0: \quad \Theta = -1 \quad \text{versus}$$
$$\mathcal{H}_1: \quad \Theta = 1$$

that minimizes the average error probability.