1. Recall that, if $F$ is a class of measurement models and $P$ a class of prior distributions, then $P$ is *conjugate* for $F$ if $\pi(\theta) \in P$ and $f_{X\,|\,\Theta}(x\,|\,\theta) \in F$ implies $f_{\Theta\,|\,X}(\theta\,|\,x) \in P$.

If $P$ is a family of probability density functions (pdfs), define the family of *$M$-term mixture densities based on $P$*, denoted $P_{\text{mix}}$ as the set of all densities of the form

$$f_\Theta(\theta) = \sum_{i=1}^{M} p_i\, f_i(\theta)$$

where $f_i(\theta) \in P$ and $p_i$, $i = 1, 2, \ldots, M$ are weights satisfying

$$p_i \geq 0, \quad \sum_{i=1}^{M} p_i = 1.$$

**(a)** Show that, if $P$ is conjugate for $F$, then $P_{\text{mix}}$ is conjugate for $F$. *Hint:* Be careful, the mixture proportions will change, i.e. they are different in the prior and posterior mixtures.

**(b)** Now take $M = 2$ and consider the model

$$X = \Theta + W$$

where

$$W \sim \mathcal{N}(0, \sigma^2)$$

and

$$\Theta \sim p\,\mathcal{N}(\theta\,|\,0, \tau_1^2) + (1-p)\,\mathcal{N}(\theta\,|\,0, \tau_2^2).$$

Assume that $\sigma^2, \tau_1^2, \tau_2^2$, and $p$ are known. Find the posterior pdf $f_{\Theta\,|\,X}(\theta\,|\,x)$.

2. ***Neural net.*** Consider the signal-plus-noise model

$$X = \Theta + W$$

where the prior pdf for the signal $\Theta$ is

$$\Theta \sim \mathrm{U}(-1, 1)$$

and noise $W$ is Gaussian

$$W \sim \mathcal{N}(0,1).$$

The signal $\Theta$ and noise $W$ are independent.

**(a)** Find the function $g(x)$ that minimizes

$$\text{BMSE} = \text{E}_{\Theta,X}\{[\text{sgn}(\Theta) - g(X)]^2\}$$

where $\text{sgn}(\cdot)$ is the *signum function*:

$$\text{sgn}(x) = \begin{cases} -1, & x \leq 0 \\ 1, & x > 0 \end{cases}.$$

**(b)** Plot $g(x)$ versus $x$.

*Comment:* Utilize $\Phi(\cdot)$, the cumulative distribution function (cdf) of a standard normal RV; see also handout # 3 from EE 420 handouts posted on `WebCT` (folder Course Readings) to refresh memory on $\Phi(\cdot)$. You may also express your solution in terms of the complementary standard-normal cdf $Q(z) = \Phi(-z)$. Both functions are available in MATLAB.

3. ***Additive shot-noise channel.*** Consider the signal-plus-noise model

$$X = \Theta + W$$

where the signal $\Theta$ follows

$$\Theta \sim \text{U}(0,1)$$

and noise $W$ is conditionally Gaussian given $\Theta = \theta$, with variance proportional to the signal, i.e.

$$\{W \mid \Theta = \theta\} \sim \mathcal{N}(0, c\,\theta)$$

for some known constant $c > 0$. Find the LMMSE estimate of $\Theta$ based on $X$. Your answer should be in terms of $X$ and $c$ only.

4. ***Noise cancellation***. A typical problem in statistical signal processing involves estimating a weak signal (e.g. the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations: one with the weak signal present and one without (by placing one microphone on the mother's belly and another close to her heart). The observations can then be combined to estimate the weak signal by "canceling out" the interference. The following is a simple version of this application. Assume that a weak signal $\Theta$ is a zero-mean random variable with variance $\tau^2$ and model the observations as follows:

$$X_1 = \Theta + W_1$$

and

$$X_2 = W_1 + W_2$$

where $W_1$ is the strong interference and $W_2$ is measurement noise. Assume that $W_1$ and $W_2$ are zero-mean random variables with known variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Further assume that $\Theta, W_1$, and $W_2$ are *uncorrelated*. Find the LMMSE estimate of $\Theta$ based on $X_1$ and $X_2$ and the corresponding BMSE. Interpret the results.

5. Define a truncated-Gaussian pdf (to non-negative values) with parameters $\mu$ and $\sigma$:

$$\mathcal{N}_{\rm t}(\theta \,|\, \mu, \tau^2) = \frac{\mathcal{N}(\theta \,|\, \mu, \tau^2)}{\Phi(\mu/\tau)} \cdot i_{[0,\infty)}(\theta)$$

where $\tau = \sqrt{\tau^2}$ and $\Phi(\cdot)$ denotes the cdf of the standard normal random variable. Check that $\mathcal{N}_{\rm t}(\theta \,|\, \mu, \tau^2)$ integrates to one with respect to $\theta$.

Consider the signal-plus-noise model

$$X[n] = \Theta[n] + W[n]$$

where $W[n]$, $n = 0, 1, \ldots, N-1$ are i.i.d. Gaussian random variables following

$$W[n] \sim \mathcal{N}(0, \sigma^2)$$

and $\Theta[n]$, $n = 0, 1, \ldots, N-1$ are conditionally i.i.d. given hyperparameters $\mu$ and $\tau^2$, following

$$\{\Theta[n] \mid \mu, \tau^2\} = \mathcal{N}_{\mathrm{t}}(\theta[n] \mid \mu, \sigma^2)$$

i.e.

$$f_{\boldsymbol{\Theta} \mid \mu, \tau^2}(\boldsymbol{\theta} \mid \mu, \tau^2) = \prod_{n=0}^{N-1} \mathcal{N}_{\mathrm{t}}(\theta[n] \mid \mu, \tau^2)$$

where $\boldsymbol{\theta} = [\theta[0], \theta[1], \ldots, \theta[N-1]]^T$. We assume that the hyperparameters $\mu$ and $\tau^2$ are independent *a priori*:

$$\pi(\mu, \tau^2) = \pi(\mu)\, \pi(\tau^2)$$

and specify the following prior pdfs for them:

$$\pi(\mu) = \mathrm{U}(\mu \mid 0, \mu_{\mathrm{MAX}}), \quad \pi(\tau^2) = \mathrm{U}(\tau^2 \mid 0, \tau^2_{\mathrm{MAX}})$$

where $\mu_{\mathrm{MAX}}$ and $\tau^2_{\mathrm{MAX}}$ are known constants.

Define $\boldsymbol{x} = [x[0], x[1], \ldots, x[N-1]]^T$. Determine the kernel of the posterior pdf

$$f_{\mu, \tau^2 \mid \boldsymbol{X}}(\mu, \tau^2 \mid \boldsymbol{x})$$

i.e. find a closed-form of this pdf up to a normalizing constant. Hence, your final answer should be

| |
|---|
| $f_{\mu, \tau^2 \mid \boldsymbol{X}}(\mu, \tau^2 \mid \boldsymbol{x}) \propto$ an algebraic expression with no integrals |

*Hint:* Apply the algebraic-integration trick presented in handout # 4c. First write out

$$f_{\mu, \tau^2 \mid \boldsymbol{X}}(\mu, \tau^2 \mid \boldsymbol{x})$$

then identify $f_{\boldsymbol{\Theta} \mid \mu, \tau^2, \boldsymbol{X}}(\boldsymbol{\theta} \mid \mu, \tau^2, \boldsymbol{x})$ and, finally, apply

$$f_{\mu, \tau^2 \mid \boldsymbol{X}}(\mu, \tau^2 \mid \boldsymbol{x}) = \frac{f_{\boldsymbol{\Theta}, \mu, \tau^2 \mid \boldsymbol{X}}(\boldsymbol{\theta}, \mu, \tau^2 \mid \boldsymbol{x})}{f_{\boldsymbol{\Theta} \mid \mu, \tau^2, \boldsymbol{X}}(\boldsymbol{\theta} \mid \mu, \tau^2, \boldsymbol{x})}.$$