Using Nonlinear Devices in Electronic Circuits

Operating Points, Small-Signal Modeling, Linear Amplifiers
Analyzing Nonlinear Circuits: Method #3

- If the variations in the voltage and current levels in the devices are very small → Small-signal assumption
- Solve for the DC quiescent point of all the nonlinear devices in the circuit using the nonlinear equations of each device
  - If the devices have piecewise models that correspond to different regions of operation, then it may be necessary to first guess the region of operation before obtaining
- Based on the DC quiescent point of each nonlinear device, develop an AC small-signal linear model for the device
- Replace each nonlinear device in the circuit with its AC small-signal linear model to obtain a linear circuit that describes the original circuit under small-signal conditions
- Use KVL, KCL and ALL LINEAR circuit analysis techniques to solve the resultant linear circuit and obtain the relationship between the AC small-signal parameters of interest
- Add the results to the DC quiescent point to obtain the full solution
Analyzing Nonlinear Circuits: Method # 3

Nonlinear Network

DC nonlinear equivalent network

Q-point

Values for small-signal parameters

AC small-signal (linear) equivalent network

AC Small-signal output

Total output

(good approximation)
What are the AC small-signal models of transistors and diodes?
Let’s consider a generic nonlinear 4-terminal device where one terminal is the reference. 3-port network

\[
\begin{align*}
    I_1 &= f_1(V_1, V_2, V_3) \\
    I_2 &= f_2(V_1, V_2, V_3) \\
    I_3 &= f_3(V_1, V_2, V_3)
\end{align*}
\]

Define

\[
\begin{align*}
    i_1 &= I_1 - I_{1Q} \\
    i_2 &= I_2 - I_{2Q} \\
    i_3 &= I_3 - I_{3Q}
\end{align*}
\]

\[
\begin{align*}
    v_1 &= V_1 - V_{1Q} \\
    v_2 &= V_2 - V_{2Q} \\
    v_3 &= V_3 - V_{3Q}
\end{align*}
\]

The AC small-signal variations in the voltages and currents are represented with small letter italic symbols

\[
\begin{align*}
    i_1 &= g_1(v_1, v_2, v_3) \\
    i_2 &= g_2(v_1, v_2, v_3) \\
    i_3 &= g_3(v_1, v_2, v_3)
\end{align*}
\]

\(f_1, f_2,\) and \(f_3\) are nonlinear functions
\(g_1,\) \(g_2,\) and \(g_3\) are linear functions
AC Small-Signal Modeling of 4-Terminal Devices

- Using the multivariate Taylor's series expansion around the quiescent point

\[
\begin{aligned}
I_1 &= f_1(V_1, V_2, V_3) \\
I_2 &= f_2(V_1, V_2, V_3) \\
I_3 &= f_3(V_1, V_2, V_3)
\end{aligned}
\]

Define \( \overline{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix} \)

If \( I_1 = f_1(V_1, V_2, V_3) \), then \( I_1 \) can be represented using the multivariate Taylor Series Expansion as

\[
I_1 = f_1(\overline{V}_Q) + \sum_{k=1}^{3} \left( \frac{\partial f_1(V_1, V_2, V_3)}{\partial V_k} \right)_{V=\overline{V}_Q} (V_k - V_{kQ}) + \ldots (H.O.T)
\]

\[
= f_1(\overline{V}_Q) \frac{\partial f_1}{\partial V_1} (V_1 - V_{1Q}) + \frac{\partial f_1}{\partial V_2} (V_2 - V_{2Q}) + \frac{\partial f_1}{\partial V_3} (V_3 - V_{3Q})
\]

note that \( I_{1Q} = f_1(\overline{V}_Q) \) and that \( i_1 = I_1 - I_{1Q} \) and \( v_k = V_k - V_{kQ} \)

\[
i_1 = g_1(v_1, v_2, v_3) = \frac{\partial f_1}{\partial V_1} (V_1 - V_{1Q}) \cdot v_1 + \frac{\partial f_1}{\partial V_2} (V_2 - V_{2Q}) \cdot v_2 + \frac{\partial f_1}{\partial V_3} (V_3 - V_{3Q}) \cdot v_3
\]
AC Small-Signal Modeling of 4-Terminal Devices

Using the multivariate Taylor’s series expansion around the quiescent point

\[ l_1 = f_1(V_1, V_2, V_3) \]
\[ l_2 = f_2(V_1, V_2, V_3) \]
\[ l_3 = f_3(V_1, V_2, V_3) \]

\[ \begin{align*}
 l_{1Q} &= f_1(V_Q) \\
 l_{2Q} &= f_2(V_Q) \\
 l_{3Q} &= f_3(V_Q)
\end{align*} \]

Define \( V_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix} \)

\[ i_1 = g_1(v_1, v_2, v_3) = \frac{\partial f_1}{\partial V_1} \bigg|_{V=V_Q} \cdot v_1 + \frac{\partial f_1}{\partial V_2} \bigg|_{V=V_Q} \cdot v_2 + \frac{\partial f_1}{\partial V_3} \bigg|_{V=V_Q} \cdot v_3 \]

\[ i_2 = g_2(v_1, v_2, v_3) = \frac{\partial f_2}{\partial V_1} \bigg|_{V=V_Q} \cdot v_1 + \frac{\partial f_2}{\partial V_2} \bigg|_{V=V_Q} \cdot v_2 + \frac{\partial f_2}{\partial V_3} \bigg|_{V=V_Q} \cdot v_3 \]

\[ i_3 = g_3(v_1, v_2, v_3) = \frac{\partial f_3}{\partial V_1} \bigg|_{V=V_Q} \cdot v_1 + \frac{\partial f_3}{\partial V_2} \bigg|_{V=V_Q} \cdot v_2 + \frac{\partial f_3}{\partial V_3} \bigg|_{V=V_Q} \cdot v_3 \]
**AC Small-Signal Modeling of 4-Terminal Devices**

- Using the multivariate Taylor’s series expansion around the quiescent point

\[
\begin{align*}
l_1 &= f_1 (V_1, V_2, V_3) \\
l_2 &= f_2 (V_1, V_2, V_3) \\
l_3 &= f_3 (V_1, V_2, V_3)
\end{align*}
\]

Define \( \bar{V}_Q = \begin{bmatrix} V_{1Q} \\ V_{2Q} \\ V_{3Q} \end{bmatrix} \) →

\[
\begin{align*}
l_{1Q} &= f_1 \left( \bar{V}_Q \right) \\
l_{2Q} &= f_2 \left( \bar{V}_Q \right) \\
l_{3Q} &= f_3 \left( \bar{V}_Q \right)
\end{align*}
\]

Let us define: \( y_{ij} = \left. \frac{\partial f_i}{\partial V_j} \right|_{\bar{V} = \bar{V}_Q} \)

\[
\begin{align*}
y_{11} &= \left. \frac{\partial f_1}{\partial V_1} \right|_{\bar{V} = \bar{V}_Q} \\
y_{12} &= \left. \frac{\partial f_1}{\partial V_2} \right|_{\bar{V} = \bar{V}_Q} \\
y_{13} &= \left. \frac{\partial f_1}{\partial V_3} \right|_{\bar{V} = \bar{V}_Q}
\end{align*}
\]

\[
\begin{align*}
y_{21} &= \left. \frac{\partial f_2}{\partial V_1} \right|_{\bar{V} = \bar{V}_Q} \\
y_{22} &= \left. \frac{\partial f_2}{\partial V_2} \right|_{\bar{V} = \bar{V}_Q} \\
y_{23} &= \left. \frac{\partial f_2}{\partial V_3} \right|_{\bar{V} = \bar{V}_Q}
\end{align*}
\]

\[
\begin{align*}
y_{31} &= \left. \frac{\partial f_3}{\partial V_1} \right|_{\bar{V} = \bar{V}_Q} \\
y_{32} &= \left. \frac{\partial f_3}{\partial V_2} \right|_{\bar{V} = \bar{V}_Q} \\
y_{33} &= \left. \frac{\partial f_3}{\partial V_3} \right|_{\bar{V} = \bar{V}_Q}
\end{align*}
\]
AC Small-Signal Modeling of Nonlinear Devices

- A generic nonlinear 4-terminal device can be represented mathematically as a 3-port linear network in small-signal conditions.
- The linear coefficients are called the Y-parameters of the network.
- The Y-parameters are Q-point dependent.
- 4-terminal device $\rightarrow$ 3-port network $\rightarrow$ 9 small-signal parameters.

\[
\begin{align*}
  i_1 &= y_{11} \cdot v_1 + y_{12} \cdot v_2 + y_{13} \cdot v_3 \\
  i_2 &= y_{21} \cdot v_1 + y_{22} \cdot v_2 + y_{23} \cdot v_3 \\
  i_3 &= y_{31} \cdot v_1 + y_{32} \cdot v_2 + y_{33} \cdot v_3
\end{align*}
\]
A generic nonlinear 4-terminal device can be represented as a linear 3-port equivalent network in small-signal conditions using linear circuit elements.

\[ y_{ij} = \frac{\partial f_i}{\partial V_j} \bigg|_{V=V_Q} \]

\[ i_1 = y_{11} \cdot v_1 + y_{12} \cdot v_2 + y_{13} \cdot v_3 \]

\[ i_2 = y_{21} \cdot v_1 + y_{22} \cdot v_2 + y_{23} \cdot v_3 \]

\[ i_3 = y_{31} \cdot v_1 + y_{32} \cdot v_2 + y_{33} \cdot v_3 \]
AC Small-Signal Modeling of 3-Terminal Devices

- A generic nonlinear 3-terminal device can be represented as a linear 2-port equivalent network in small-signal conditions using linear circuit elements.

\[ y_{ij} = \frac{\partial f_i}{\partial V_j} \bigg|_{\vec{V}=\vec{V}_Q} \]

\[ i_1 = y_{11} \cdot v_1 + y_{12} \cdot v_2 + y_{13} \cdot v_3 \]
\[ i_2 = y_{21} \cdot v_1 + y_{22} \cdot v_2 + y_{23} \cdot v_3 \]
\[ i_3 = y_{31} \cdot v_1 + y_{32} \cdot v_2 + y_{33} \cdot v_3 \]
AC Small-Signal Modeling of 3-Terminal Devices

- A generic nonlinear 3-terminal device can be represented as a linear 2-port equivalent network in small-signal conditions using linear circuit elements.
- 3-terminal device $\rightarrow$ 2-port network $\rightarrow$ 4 small-signal parameters

\[
\begin{align*}
\mathbf{V}_Q &= \begin{bmatrix} V_{1Q} \\ V_{2Q} \end{bmatrix} \\
\mathbf{l}_Q &= \begin{bmatrix} l_{1Q} \\ l_{2Q} \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{V}_Q) \\ f_2(\mathbf{V}_Q) \end{bmatrix} \\
\mathbf{y}_{ij} &= \frac{\partial f_i}{\partial V_j} \bigg|_{\mathbf{V} = \mathbf{V}_Q}
\end{align*}
\]

\[
\begin{align*}
i_1 &= y_{11} \cdot V_1 + y_{12} \cdot V_2 \\
i_2 &= y_{21} \cdot V_1 + y_{22} \cdot V_2
\end{align*}
\]
AC Small-Signal Modeling of 2-Terminal Devices

- A generic nonlinear 2-terminal device can be represented as a linear 1-port equivalent network in small-signal conditions using linear circuit elements.

\[ y_{ij} = \frac{\partial f_i}{\partial V_j} \bigg|_{V=V_Q} \]

\[
\begin{align*}
  i_1 &= y_{11} \cdot v_1 + y_{12} \cdot v_2 + y_{13} \cdot v_3 \\
  i_2 &= y_{21} \cdot v_1 + y_{22} \cdot v_2 + y_{23} \cdot v_3 \\
  i_3 &= y_{31} \cdot v_1 + y_{32} \cdot v_2 + y_{33} \cdot v_3
\end{align*}
\]
AC Small-Signal Modeling of 2-Terminal Devices

- A generic nonlinear 2-terminal device can be represented as a linear 1-port equivalent network in small-signal conditions using linear circuit elements.
- 2-terminal device → 1-port network → 1 small-signal parameters

\[ I_1 = f_1 \left( V_1 \right) \]

Define \( V_Q = V_{iQ} \) → \( I_Q = f_1 \left( V_Q \right) \)

\[ y_{ij} = \frac{\partial f_i}{\partial V_j} \bigg|_{V=V_Q} \]

\[ I_1 = y_{11} \cdot V_1 \]
AC Small-Signal Modeling of Nonlinear Devices

Nonlinear Device

Linearized Small-signal Device

Diagram showing the modeling process of nonlinear devices into linearized small-signal devices.
What is the AC small-signal equivalent of MOSFETs?

A MOSFET is actually a 4-terminal device but for many applications acceptable predictions of small-signal performance can be obtained by treating it as a 3-terminal device by assuming the bulk terminal is connected to the source.

We will start with developing the small-signal model of the 3-terminal case, then develop the 4-terminal case.
MOSFET is usually operated in saturation region in linear applications where a small-signal model is needed, so we will develop the small-signal model in the saturation region.

Large Signal Model

\[ I_D = \begin{cases} 
0 & \text{if } V_{GS} \leq V_T \\
\mu C_{OX} \frac{W}{L} \left( V_{GS} - V_T - \frac{V_{DS}}{2} \right) V_{DS} & \text{if } V_{GS} \geq V_T, V_{DS} < V_{GS} - V_T \\
\mu C_{OX} \frac{W}{2L} \left( V_{GS} - V_T \right)^2 \times (1 + \lambda V_{DS}) & \text{if } V_{GS} \geq V_T, V_{DS} \geq V_{GS} - V_T 
\end{cases} \]
AC Small-Signal Modeling of MOSFETs: 3-Terminals

3-terminal device $\rightarrow$ 2-port

$$I_G = f_1(V_{GS}, V_{DS}) = 0$$
$$I_D = f_2(V_{GS}, V_{DS}) = \mu C_{OX} \frac{W}{2L} (V_{GS} - V_T)^2 \times (1 + \lambda V_{DS})$$

Define

$$V_Q = \begin{bmatrix} V_{GSQ} \\ V_{DSQ} \end{bmatrix} \rightarrow I_{GQ} = f_1(\overline{V_Q})$$
$$I_{DQ} = f_2(\overline{V_Q})$$

$$y_{11} = \frac{\partial f_1}{\partial V_1} |_{\overline{V} = Q} = \frac{\partial I_G}{\partial V_{GS}} |_{\overline{V} = Q} = 0$$
$$y_{12} = \frac{\partial f_1}{\partial V_2} |_{\overline{V} = Q} = \frac{\partial I_G}{\partial V_{DS}} |_{\overline{V} = Q} = 0$$

$$y_{21} = \frac{\partial f_2}{\partial V_1} |_{\overline{V} = Q} = \frac{\partial I_D}{\partial V_{GS}} |_{\overline{V} = Q} = \mu C_{OX} \frac{W}{L} (V_{GSQ} - V_T) \times (1 + \lambda V_{DSQ}) \approx \mu C_{OX} \frac{W}{L} (V_{GSQ} - V_T)$$

$$y_{22} = \frac{\partial f_2}{\partial V_2} |_{\overline{V} = Q} = \frac{\partial I_D}{\partial V_{DS}} |_{\overline{V} = Q} = \lambda \times \mu C_{OX} \frac{W}{2L} (V_{GSQ} - V_T)^2 \approx \lambda \times I_{DQ}$$
AC Small-Signal Modeling of MOSFETs: 3-Terminals

Large Signal Model

\[ I_G = 0 \]
\[ I_D = \mu C_{OX} \frac{W}{2L} (V_{GS} - V_T)^2 \times (1 + \lambda V_{DS}) \]

Small Signal Model

\[ y_{11} = 0 \]
\[ y_{12} = 0 \]
\[ y_{21} \approx \mu C_{OX} \frac{W}{L} (V_{GSQ} - V_T) \]
\[ y_{22} \approx \lambda \times i_{DQ} \]

\[ i_g = y_{11} \cdot V_{gs} + y_{12} \cdot V_{ds} \]
\[ i_d = y_{21} \cdot V_{gs} + y_{22} \cdot V_{ds} \]
AC Small-Signal Modeling of MOSFETs: 3-Terminals

\[ g_m = y_{21} \approx \mu C_{ox} \frac{W}{L} (V_{GSQ} - V_T) \rightarrow \text{Transistor Transconductance} \]

\[ r_o = \frac{1}{g_o} = \frac{1}{y_{22}} \approx \frac{1}{\lambda DUQ} \rightarrow \text{Transistor Output Impedance} \]

**Diagram:**

- **Input Current:** \( i_g \)
  - **Input Voltage:** \( V_{gs} \)
  - **Output Current:** \( i_d \)
  - **Output Voltage:** \( V_{ds} \)

\[ i_g = y_{11} \cdot V_{gs} + y_{12} \cdot V_{ds} \]

\[ i_d = y_{21} \cdot V_{gs} + y_{22} \cdot V_{ds} \]

For small signal assumption:

\[ i_g = 0 \]

\[ i_d = g_m \cdot V_{gs} + g_o \cdot V_{ds} \]

- In this case, \( g_m \) is the transconductance and \( g_o \) is the transconductance gain.
AC Small-Signal Modeling of MOSFETs: 3-Terminals

\[
g_m = y_{21} \approx \mu C_{OX} \frac{W}{L} (V_{GSQ} - V_T) \rightarrow \text{Transistor Transconductance}
\]

\[
r_o = \frac{1}{g_o} = \frac{1}{y_{22}} \approx \frac{1}{\lambda I_{DQ}} \rightarrow \text{Transistor Output Impedance}
\]

Transistor Transconductance can be expressed in several ways:

\[
g_m = \mu C_{OX} \frac{W}{L} (V_{GSQ} - V_T) = \sqrt{2\mu C_{OX} \frac{W}{L}} \times I_{DQ} = \frac{2I_{DQ}}{(V_{GSQ} - V_T)}
\]